Complexity in Multi-Delay Physiological Feedback Systems

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❖ OUTLINE

1) Chaos to simple dynamics via many delays: how?

→COMPLEXITY IS NOT NECESSARILY HIGH WITH MULTIPLE DELAYS

→COMPLEXITY COLLAPSES CAN HAPPEN WITH EVEN A FEW DELAYS

2) Complexity collapse in neural nets?

→TRANSIENT CHAOS, RANDOM PERIODS OF SYNCHRONY

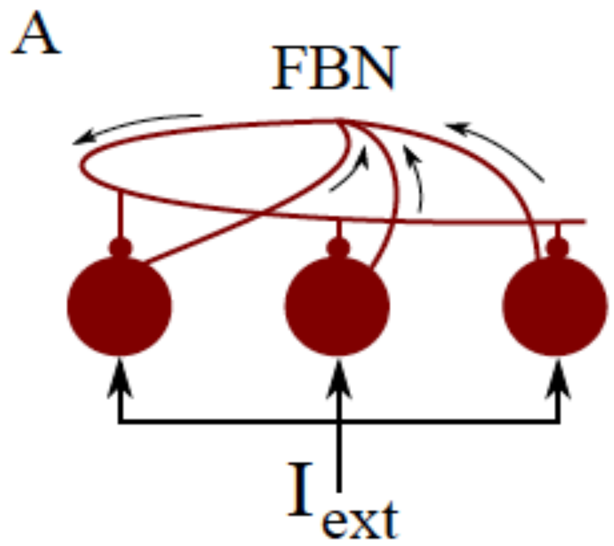
SIMPLE PUNCHLINES

- More delays does not imply more complexity

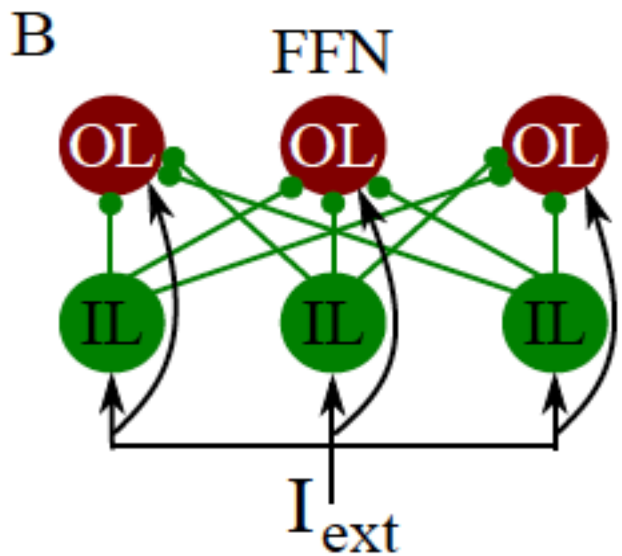
- Delayed feedback loops in physiology may be able to do novel forms of:

- Random "number" generation
- Prediction
- Deep learning

FB: DELAYED FEEDBACK FF: FEEDFORWARD+DELAY

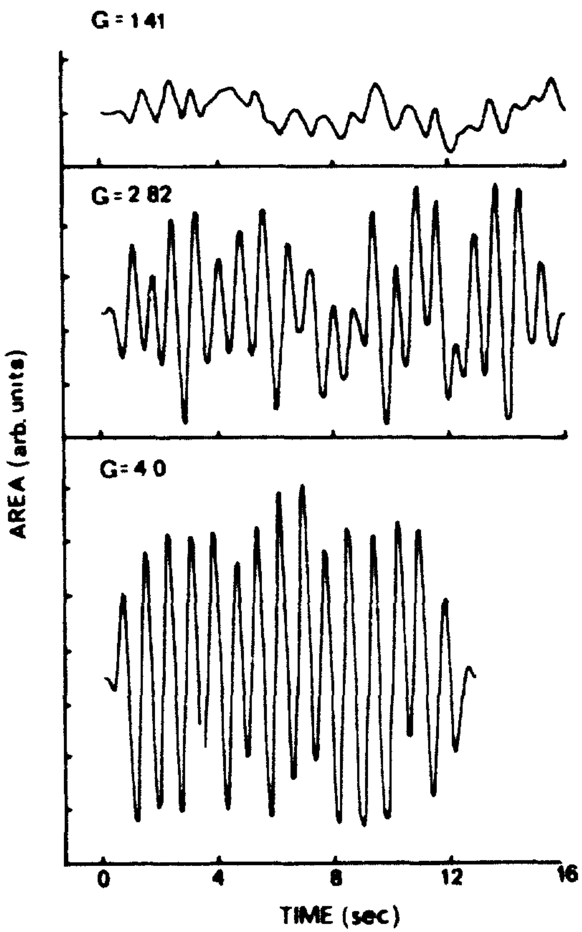


Feedback delay (INHIB)



Feedforward delay

(HUMAN) PUPIL AREA OSCILLATIONS BY CONTROLLING THE FEEDBACK GAIN

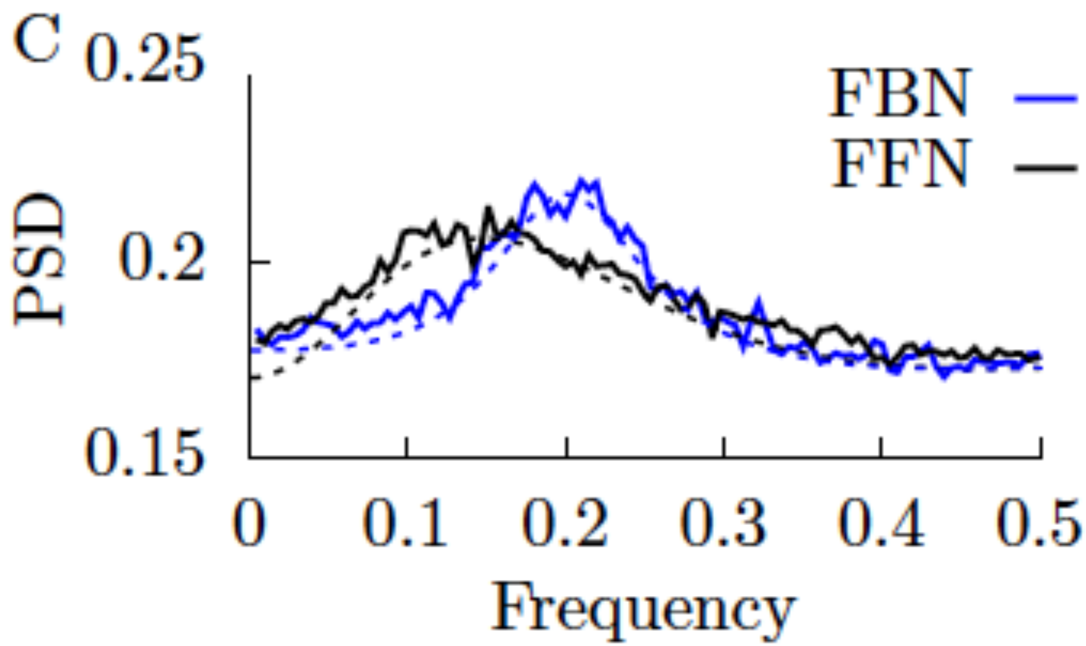


$$\frac{dA(t)}{dt} + \alpha A(t) = \frac{c\theta^n}{\theta^n + A^n(t-\tau)} + k$$

+ noise

Longtin, Milton, Bos, Mackey, Phys. Rev. A. 1990

BOTH DISPLAY A PEAK !



Many ways to get rhythms!!

Payeur & Longtin, Phys. Rev. E. 2015

- Increasing a single delay increases entropy (Farmer, Physica D 1984)

- But what about the number of delays?

Increased complexity

- Fisher et al. PRL 1994: add 2nd delay to laser system → *chaos to hyperchaos*
- Xu et al., Optics Lett. 2017: large number of random delays → *hyperchaos for random number generation*

Decreased complexity

- Ahlborn+Parlitz, PRL2004: *2nd delay on Chua circuit: chaos replaced by limit cycles*
- Mensour+Longtin, PhysLett A 1995: *controlling chaos in Mackey-Glass with a 2nd delay*

PARADOX

1) Multiple delays can increase the complexity (entropy)

2) But distributed delays have relatively lower complexity

What happens in between??

❖ SEMICONDUCTOR LASER MODEL WITH OPTICAL FEEDBACK

- Semiconductor laser with multiple optical feedbacks can be modeled with Lang-Kobayashi (LK) equations:

$$\frac{dE(t)}{dt} = (1+i\alpha)\left[\frac{G_E(N(t)-N_0)}{1+\epsilon|E(t)|^2} - \gamma_E\right]E(t) + \frac{\kappa}{M}\sum_{i=1}^{M} E(t-\tau_i)e^{-i\omega\tau_i}$$

$$\frac{dN(t)}{dt} = \gamma_N(J_r N_{th} - N(t)) - \frac{G_N(N(t)-N_0)}{1+\epsilon|E(t)|^2}|E(t)|^2$$

➢ Dynamics of LK model is governed by two main time scales:

Average delay and response time

- In distributed delay case, the state of the system depends on a continuum of past states:

$$\frac{dE}{dt} = \int_0^\infty g(\tau)F[N(t), E(t), E(t-\tau)]d\tau,$$

$$\frac{dN(t)}{dt} = \gamma_N[J_r N_{th} - N(t)] - \frac{G_N[N(t)-N_0]}{1+\epsilon|E(t)|^2}|E(t)|^2$$

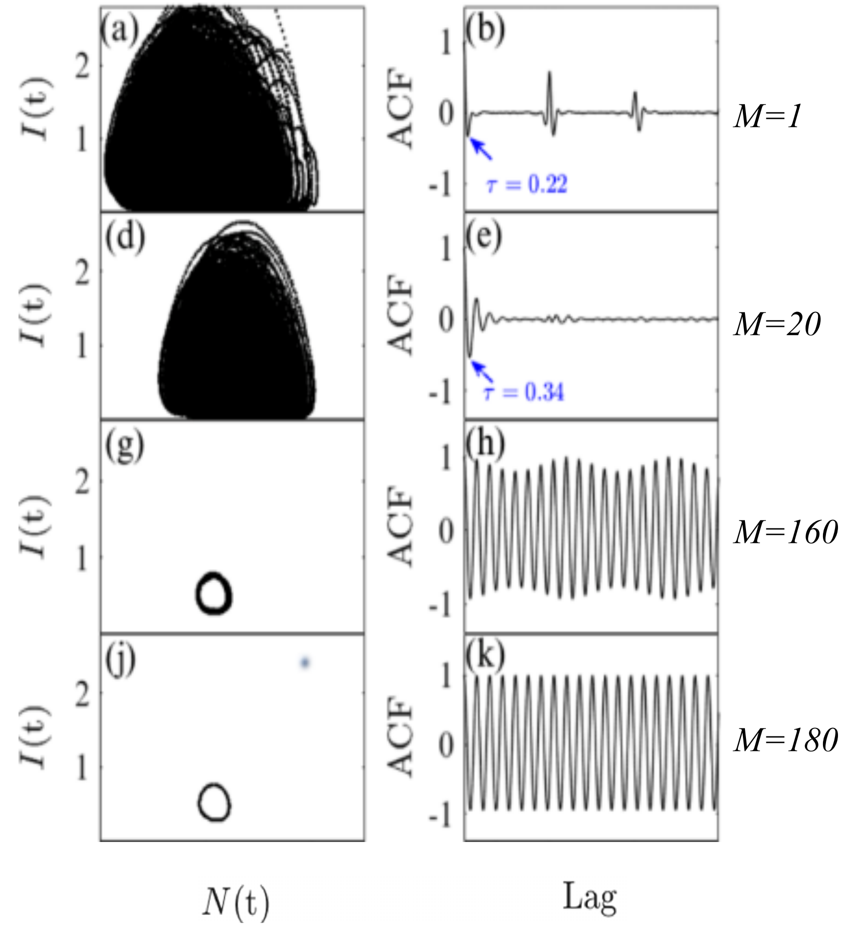
❖ SEMICONDUCTOR LASER MODEL WITH OPTICAL FEEDBACK

- Weak or moderate degree of chaos:
  Dominant time scale is the **time delay**

- High degree of chaos:
  Dominant time scale is the **relaxation oscillation period**

- Torus behavior:
  Both **relaxation oscillation period and average of the time delays govern the dynamic**

- Limit cycle:
  Simple oscillation with **relaxation oscillation period**

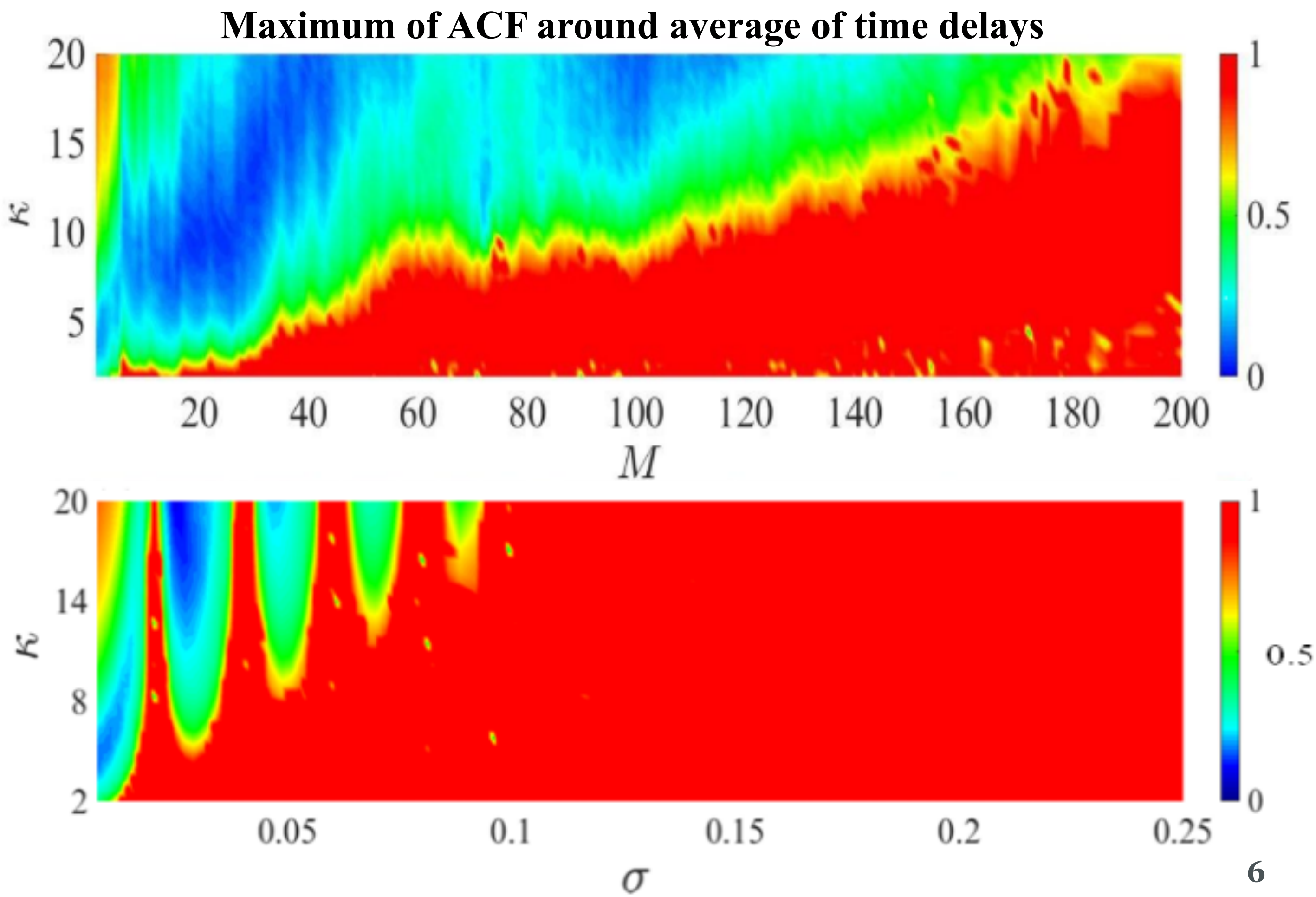


❖ MIXED EFFECTS WHEN ADDING DELAYS

- Degree of complexity via the maximum of the *autocorrelation function* (ACF) around the time delay.

Blue: chaos

Red: fixed point



S. K. Tavakoli and A. Longtin, "Multi-delay complexity collapse," Phys. Rev. Research 2, 033485 (2020)

❖ FIXED AVERAGE DELAY: *M* INDUCES BIFURCATIONS

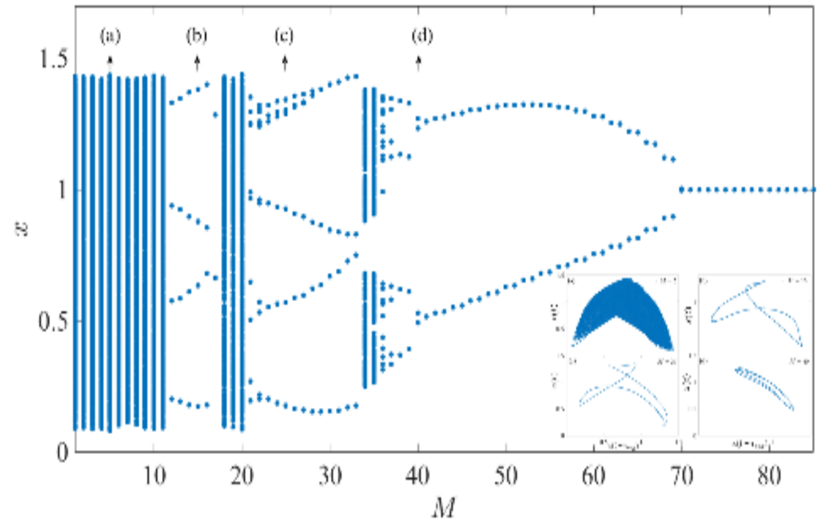
- Mackey Glass equation (standard parameters):

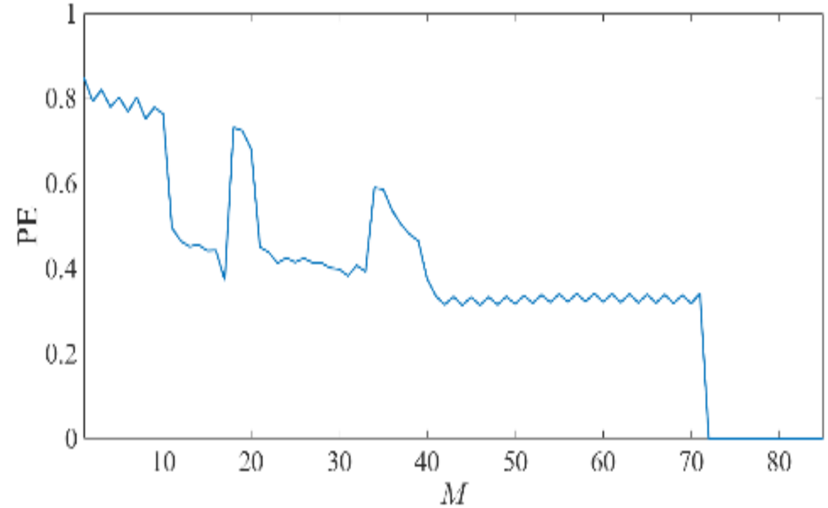
$$\frac{dx}{dt} = -x(t) + \frac{b}{M}\sum_{i=1}^{M}\frac{x(t-\tau_i)}{1+x(t-\tau_i^{10})}$$

- Delays are picked as:

$$\tau_i = \tau_{avg} - (i-1)/2\Delta\tau, \quad i \text{ odd}$$
$$\tau_i = \tau_{avg} + (i)/2\Delta\tau, \quad i \text{ even}$$

- Increasingly large periodic windows occur for increasing number of delays M, culminating in an inverse period-doubling cascade that ends with a fixed point.



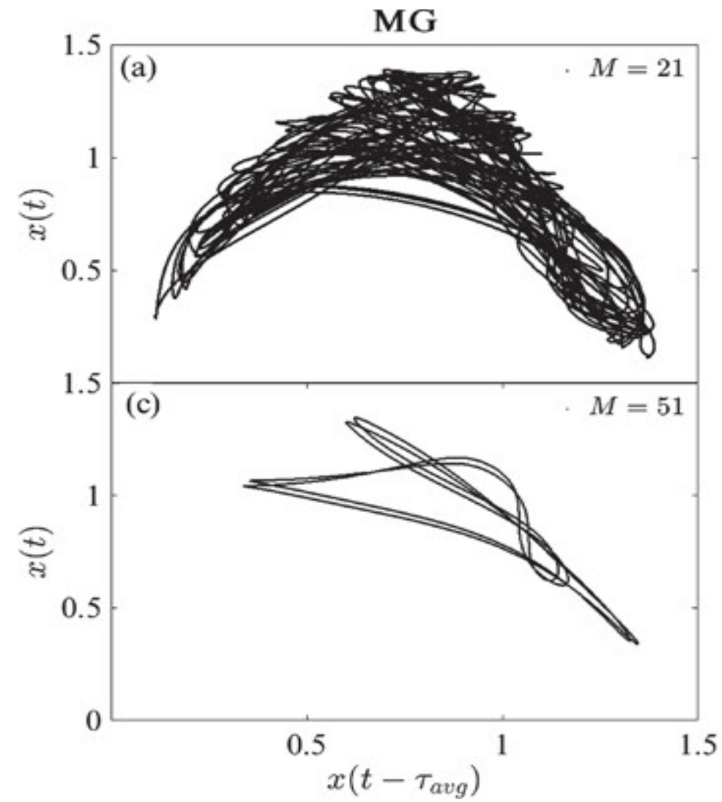


- Whenever the delay-to-response time ratio is large, first order nonlinear DDE's exhibit high-dimensional chaos and multistability.

- We use another scheme to pick delays by keeping the minimum delay fixed and increasing the average of the delays:

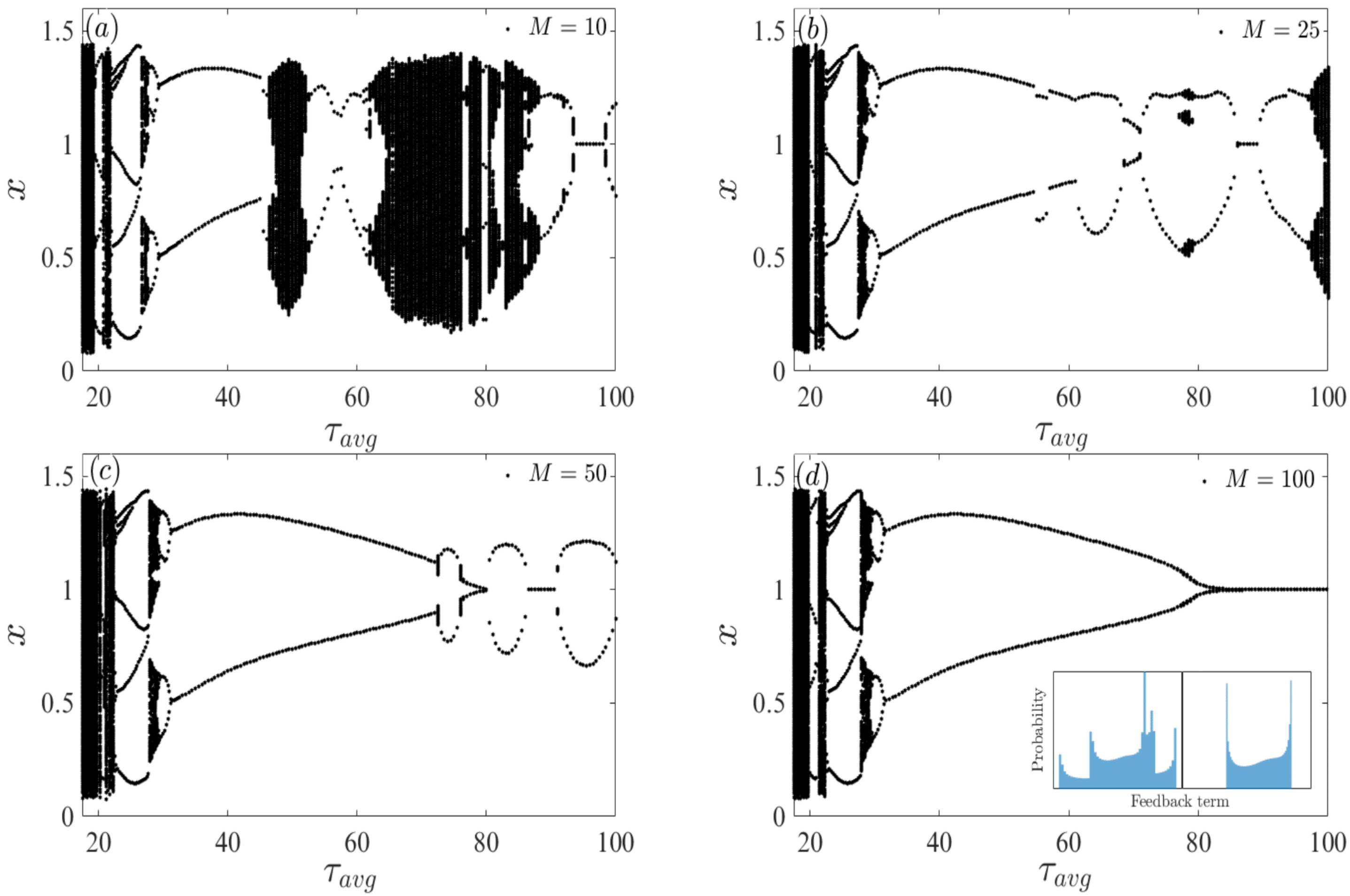
$$\tau_i = \tau_{min} + i\Delta\tau_i$$

Complexity collapse still observed when the time delays are larger than the response time.



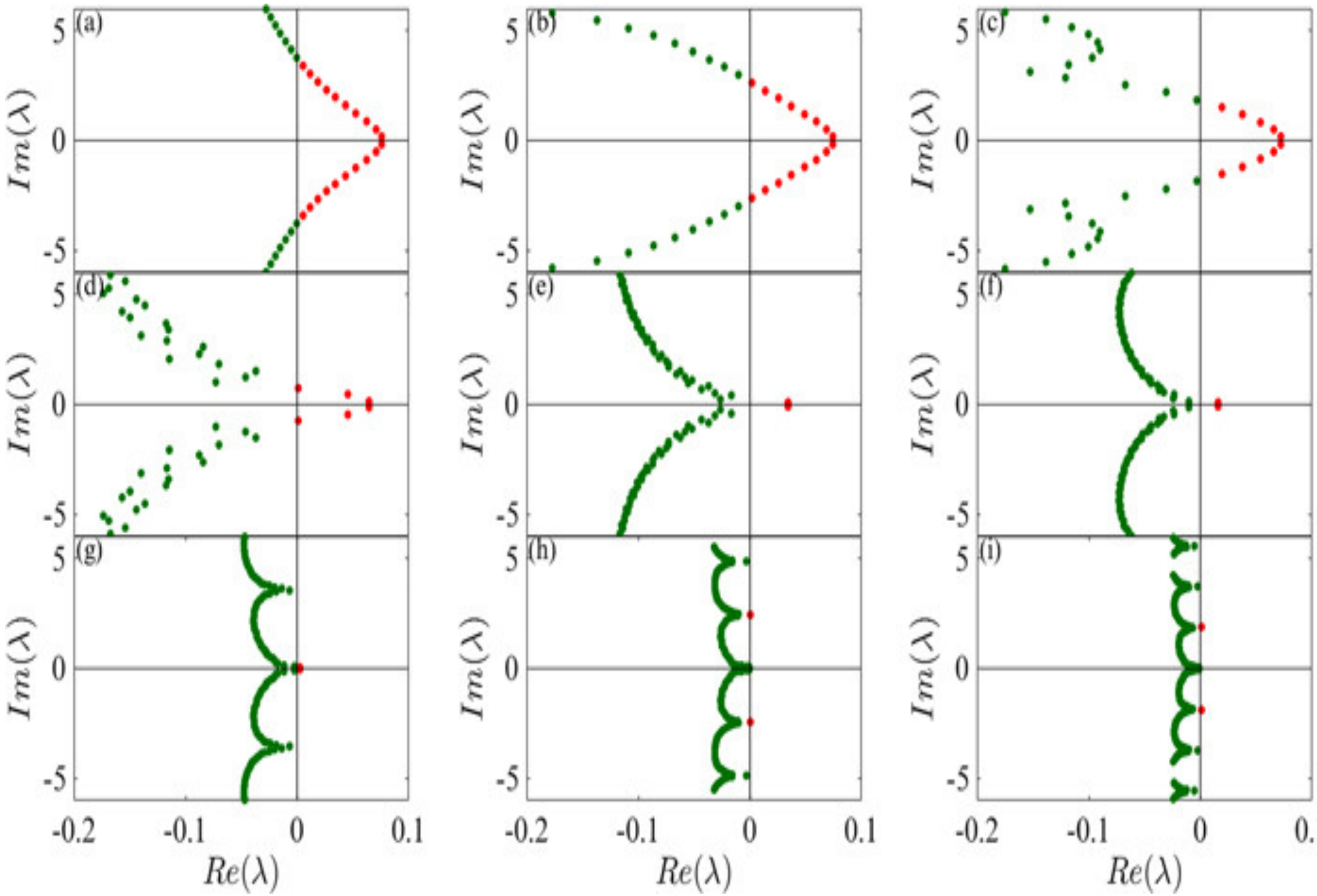
S. Kamyar Tavakoli and André Longtin, *"Dynamical invariants and inverse period-doubling cascades in multi-delay systems", Chaos 31, 103129 (2021)*

- We found that complexity collapse occurs for even few number of delays, provided there is a sufficiently large minimum delay.



- A larger number of delays favors fixed point behavior rather than limit cycles.

• Behavior of eigenvalues around fixed point when *M=50,* for increasing average delay (a) → (i)



- Characteristic equations:

$$\Delta(\lambda) = -\lambda - 1 + \frac{F'(x^*)}{M} \sum_{j=1}^{M} e^{-\lambda \tau_j}$$

$$\lambda = \mu + i\omega$$

- Separating real and imaginary part of the characteristic equation:

$$\frac{\mu+1}{F'(x^*)} = \frac{e^{-\mu\tau_{min}}}{M} \sum_{j=1}^{M} e^{-\mu(j-1)\Delta\tau} \cos(\omega(\tau_{min} + (j-1)\Delta\tau))$$

$$-\frac{\omega}{F'(x^*)} = \frac{e^{-\mu\tau_{min}}}{M} \sum_{j=1}^{M} e^{-\mu(j-1)\Delta\tau} \sin(\omega(\tau_{min} + (j-1)\Delta\tau))$$

- Putting the above equations in integral form:

$$\frac{\mu+1}{F'(x^*)} \approx \frac{e^{-\mu\tau_{min}}}{T_M + \Delta\tau} \int_0^{T_M} e^{-\mu\tau'} \cos(\omega(\tau_{min} + \tau'))d\tau'$$

$$-\frac{\omega}{F'(x^*)} \approx \frac{e^{-\mu\tau_{min}}}{T_M + \Delta\tau} \int_0^{T_M} e^{-\mu\tau'} \sin(\omega(\tau_{min} + \tau'))d\tau'$$

$$T_m = \tau_{max} - \tau_{min}$$

$$\underbrace{\frac{\mu+1}{F'(x^*)} - \frac{e^{-\mu\tau_{min}}\left[e^{-\mu T_M}\sin(\omega(\tau_{min}+T_M)) - \sin(\omega\tau_{min})\right]}{\omega T_M}}_{f(\mu,\omega)} = 0$$

$$\underbrace{\frac{-\omega}{F'(x^*)} - \frac{e^{-\mu\tau_{min}}\left[\cos(\omega\tau_{min}) - e^{-\mu T_m}\cos(\omega(\tau_{min}+T_m))\right]}{\omega T_m}}_{g(\mu,\omega)} = 0$$

❖ LYAPUNOV EXPONENTS

- Lyapunov exponents are calculated using Chebyshev polynomial nodes.
- There are m_i+1 Chebyshev nodes between two-time delays:

$$\theta_{ij} = \frac{(\tau_i - \tau_{i-1})}{2} \cos\left(\frac{j\pi}{m_i}\right) - \frac{(\tau_i + \tau_{i-1})}{2},$$

$$i = 1, \ldots, M, \quad j = 0, \ldots, m_i,$$

- Dynamical Equations to estimate Lyapunov spectrum:

$$u_0'(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_M)),$$

$$u_k'(t) = \textstyle\sum_{j=0}^{m_t} d_{kj} u_j(t) \qquad k = 1, \ldots, m_t,$$

m_t :total number of differential equations

d_{kj}: elements of Chebyshev differentiation matrix

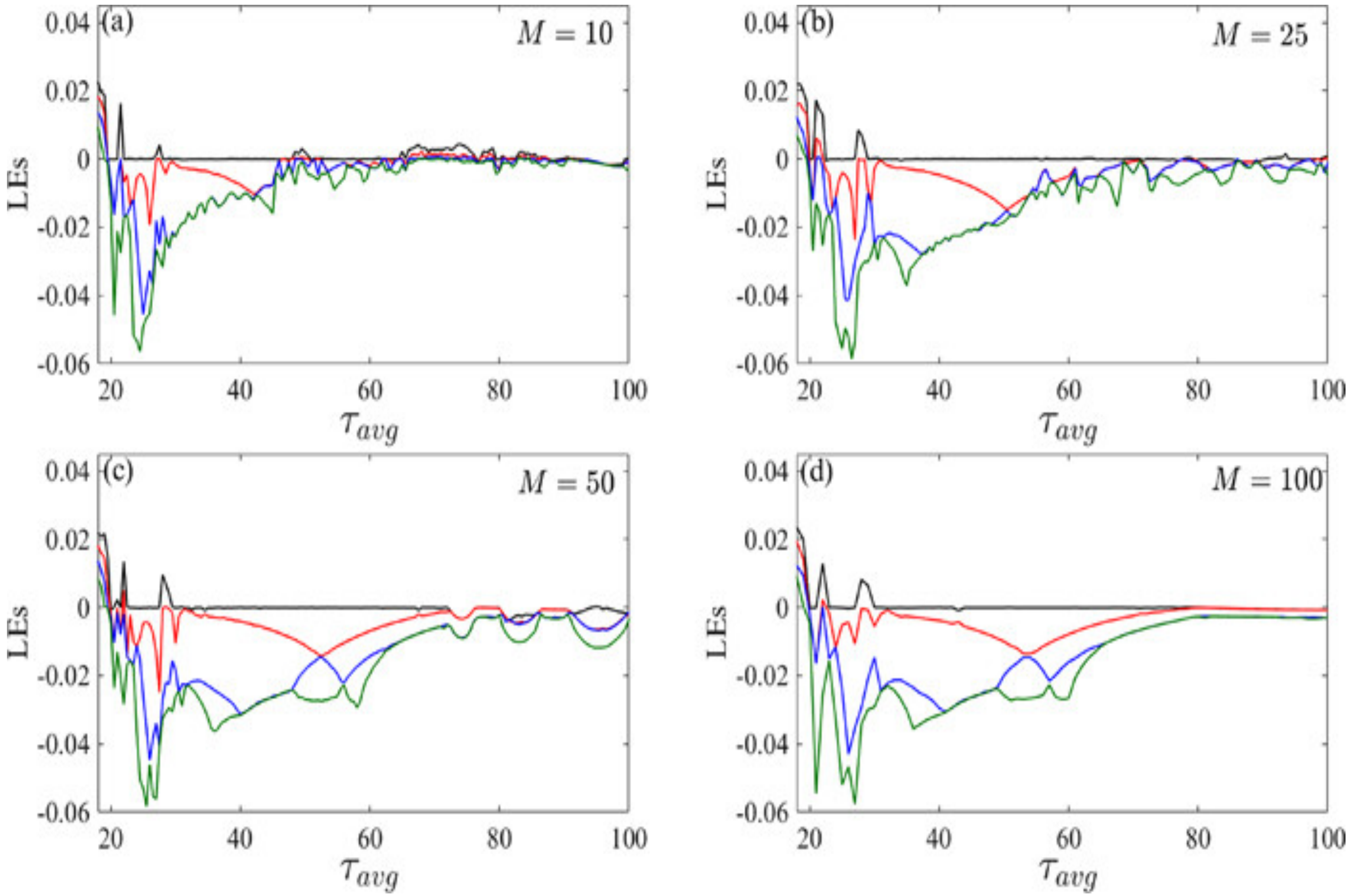
$$A = \begin{pmatrix}
\frac{df}{dx} & \cdots & \frac{df}{dx_{\tau_1}} & 0 & \cdots & 0 & \cdots & & \frac{df}{dx_{\tau_M}} \\
d_{1,0} & \cdots & d_{1,m_1} & 0 & \cdots & \vdots & \cdots & & 0 \\
\vdots & \ddots & \vdots & \vdots & \cdots & 0 & \cdots & & \vdots \\
d_{m_1,0} & \cdots & d_{m_1,m_1} & 0 & & \vdots & \cdots & & 0 \\
0 & \cdots & d_{m_1+1,m_1} & \cdots & d_{m_1+1,m_1+m_2} & 0 & \cdots & & 0 \\
\vdots & & \vdots & \ddots & \vdots & \vdots & \cdots & & \vdots \\
0 & \cdots & d_{m_1+m_2,m_1} & \cdots & d_{m_1+m_2,m_1+m_2} & 0 & \cdots & & \vdots \\
\vdots & & & & & \ddots & & & 0 \\
0 & & & & & 0 & d_{m_{tot}-m_M+1,m_{tot}-m_M} & \cdots & d_{m_{tot}-m_M+1,m_{tot}} \\
\vdots & & & & & & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & & 0 & d_{m_{tot},m_{tot}-m_M} & \cdots & d_{m_{tot},m_{tot}}
\end{pmatrix}$$

Generalization of Breda and Schiava, DCDS2018

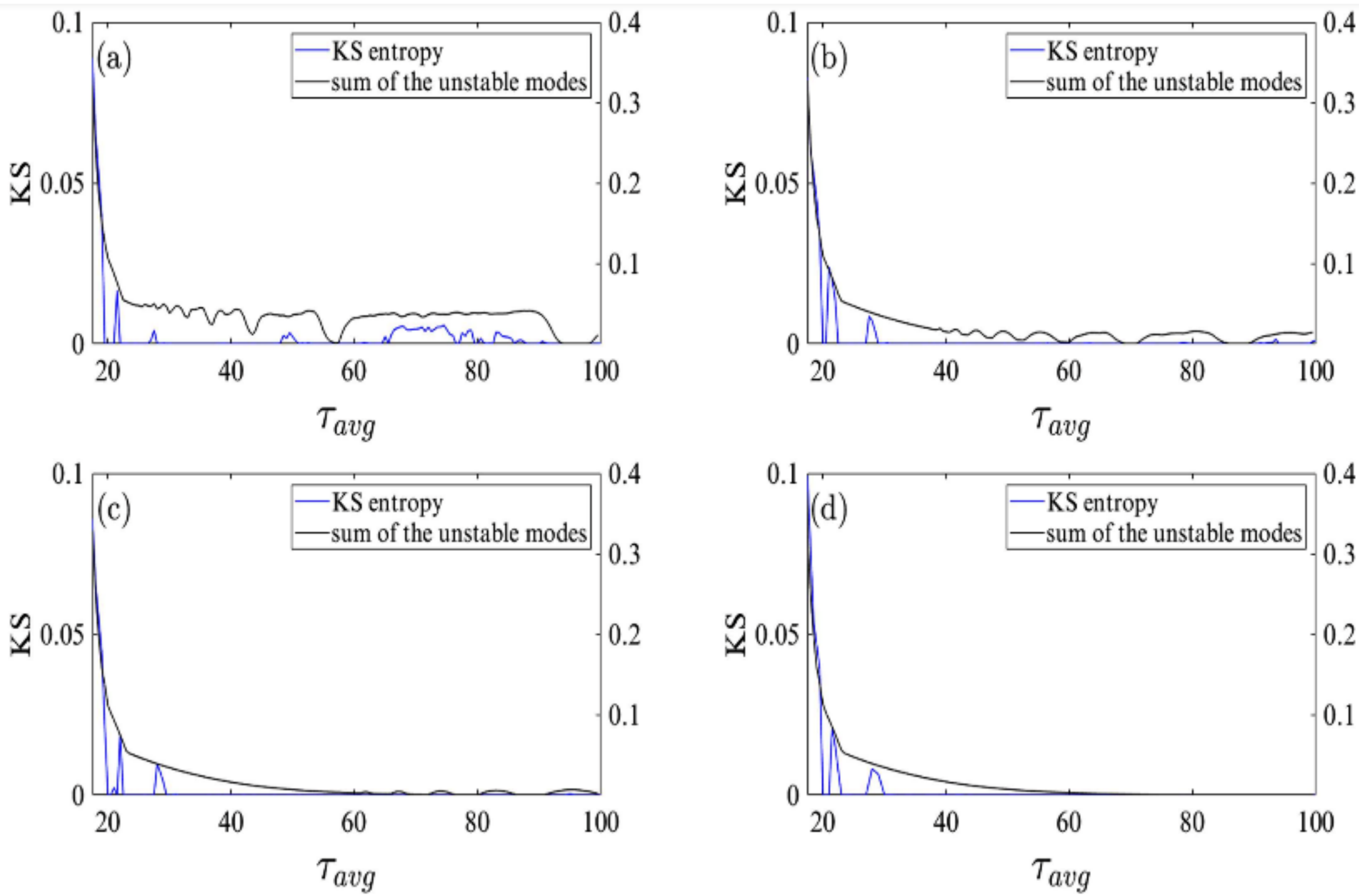
- Lyapunov exponents calculated by integrating the linearized dynamics for the u_k

$$\frac{d\delta u_i}{dt} = A\delta u_i(t)$$

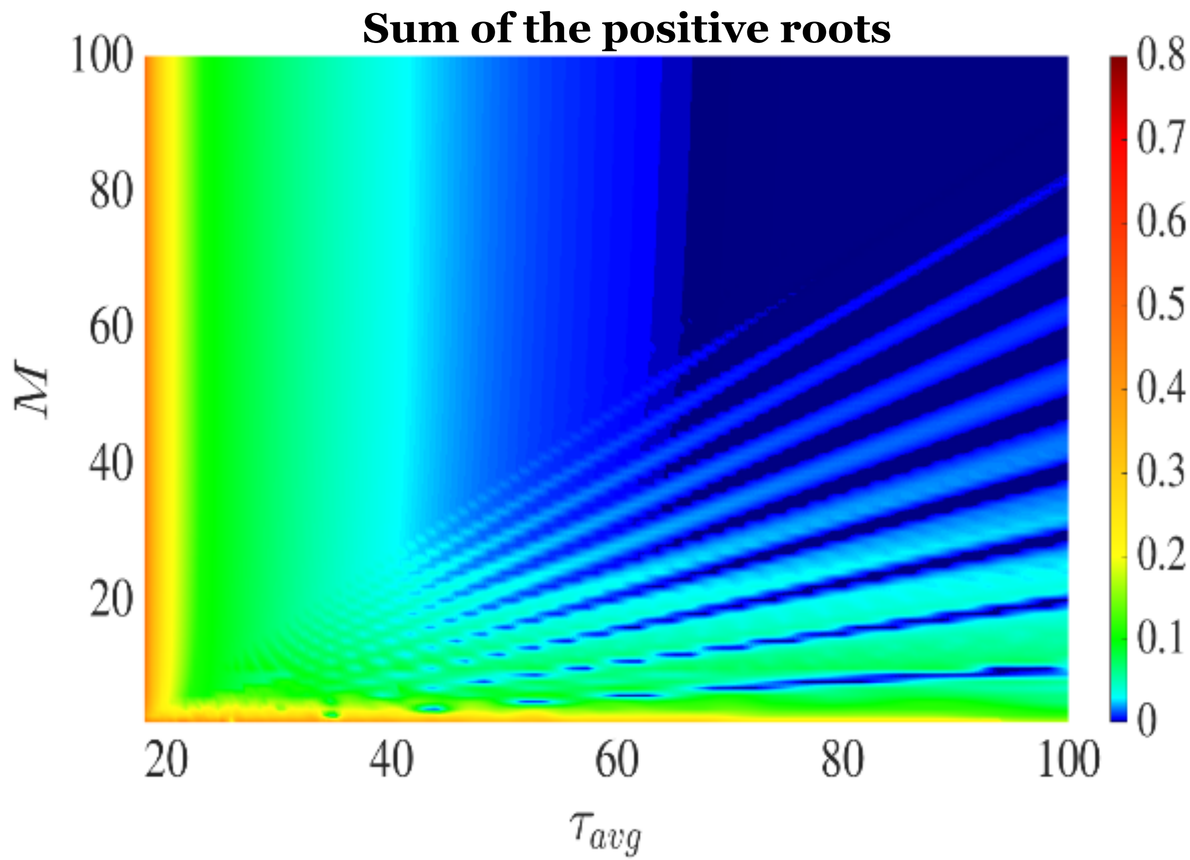
A is the Chebyshev differentiation matrix



- Comparison between Sum of Positive Lyapunov Exponents and Sum of positive part of eigenvalues



- Sum of Real(eigenvalues): proxy for solution complexity



❖ MODEL MIMICKING SPARSELY CONNECTED BRAIN CIRCUITS

- Recurrent Neural network with multiple local time delays:

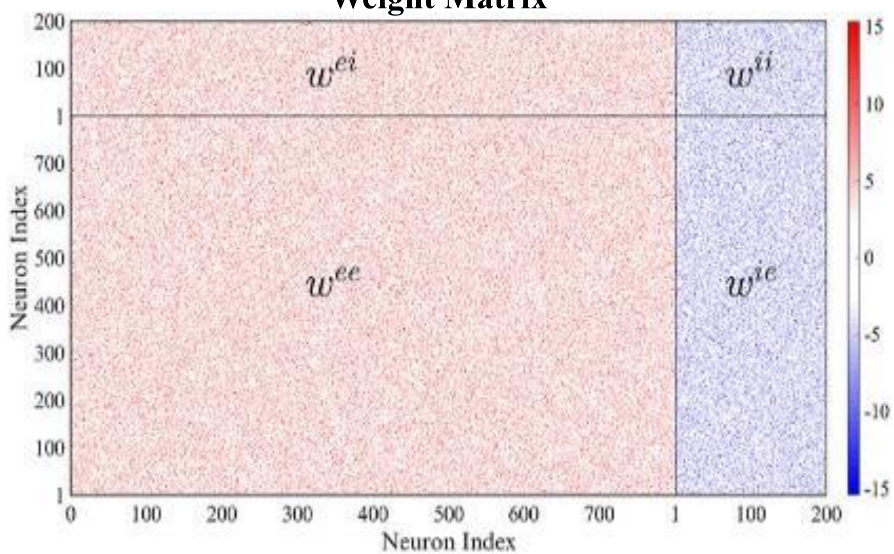
$$\alpha_e^{-1}\frac{du_j}{dt} = -u_j + \frac{1}{n_e}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_e} w_{jk}^{ee}\phi(u_k(t-\tau_l)) + \frac{1}{n_i}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_i} w_{jk}^{ie}\phi(v_k(t-\tau_l))$$

$$\alpha_i^{-1}\frac{dv_j}{dt} = -v_j + \frac{1}{n_e}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_e} w_{jk}^{ei}\phi(u_k(t-\tau_l)) + \frac{1}{n_i}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_i} w_{jk}^{ii}\phi(v_k(t-\tau_l))$$

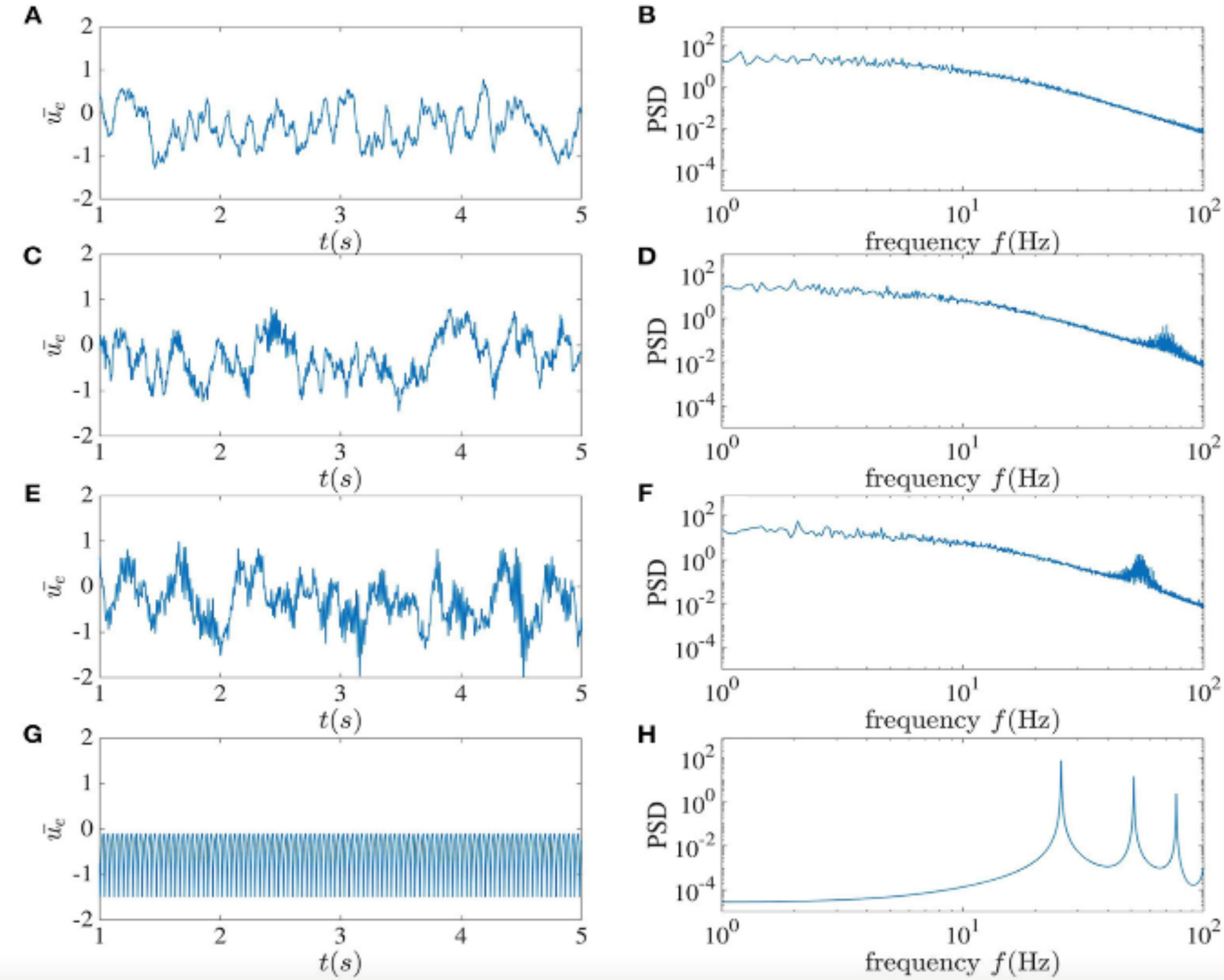
- *Rich S, Hutt A, Skinner FK, Valiante TA, Lefebvre J.. Sci Rep. (2020); 10 (1):1–17*
- *Park, S. H., Griffiths, J. D., Longtin, A., and Lefebvre, J. (2018). Front. Appl. Math.*

- We consider a recurrent local network of 80% excitatory and 20% inhibitory rate model neurons with 10% connection probability.

Weight Matrix



- Recurrent Neural network with single local time delay: Effect of increasing the delay from 2 ms to 10 ms



S. Kamyar Tavakoli and André Longtin, "Complexity Collapse, Fluctuating Synchrony, and Transient Chaos in Neural Networks With Delay Clusters", Frontiers Syst. Neurosci. 31, 103129 (2021)

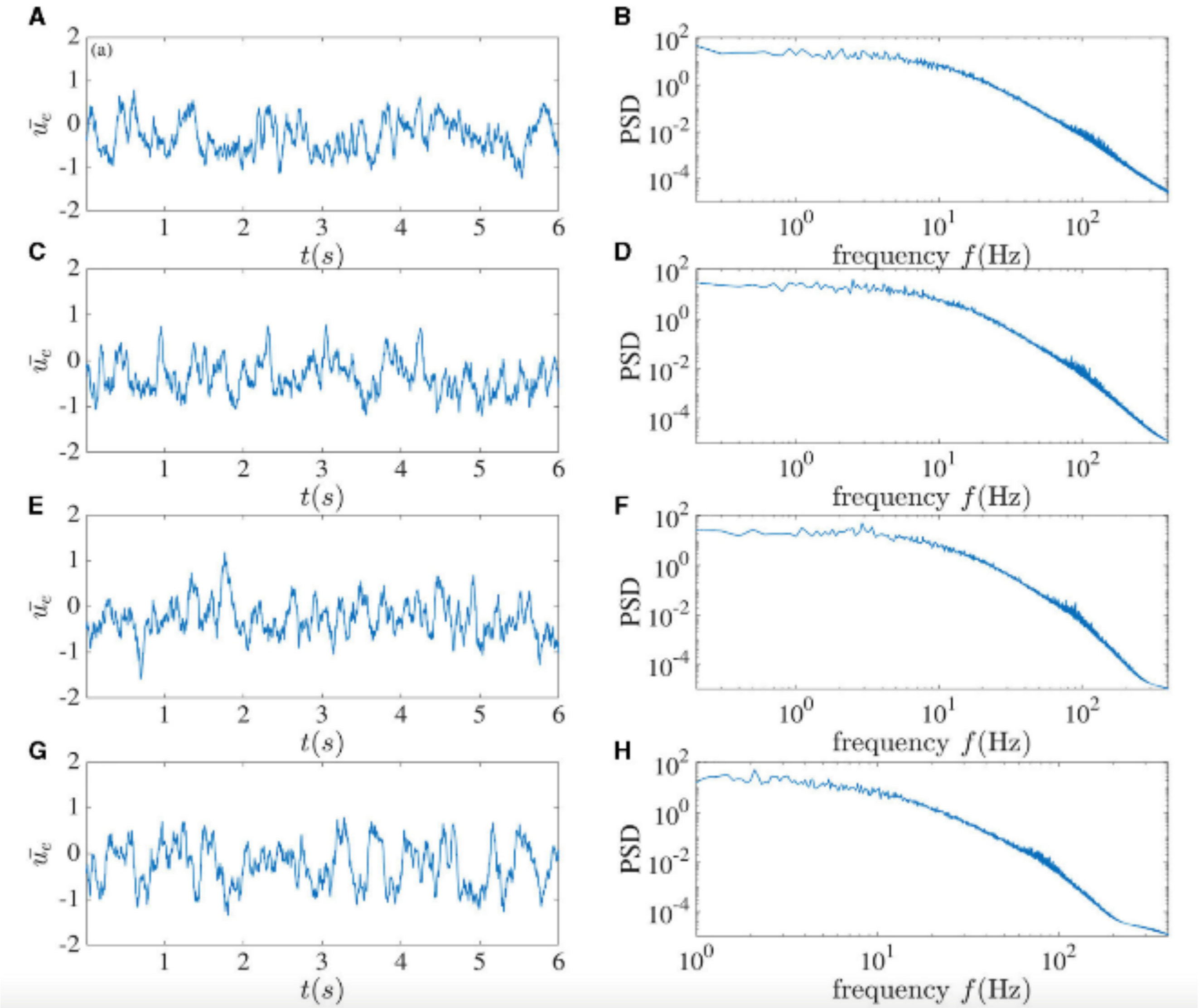
- Recurrent Neural network with local multiple time delays: no complexity collapse!

M=6

M=11

M=16

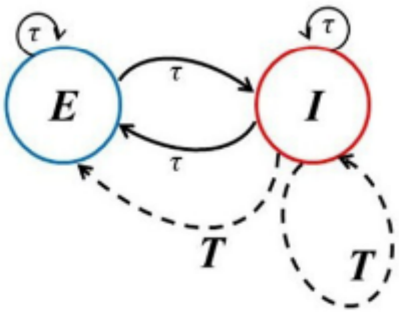
M=21



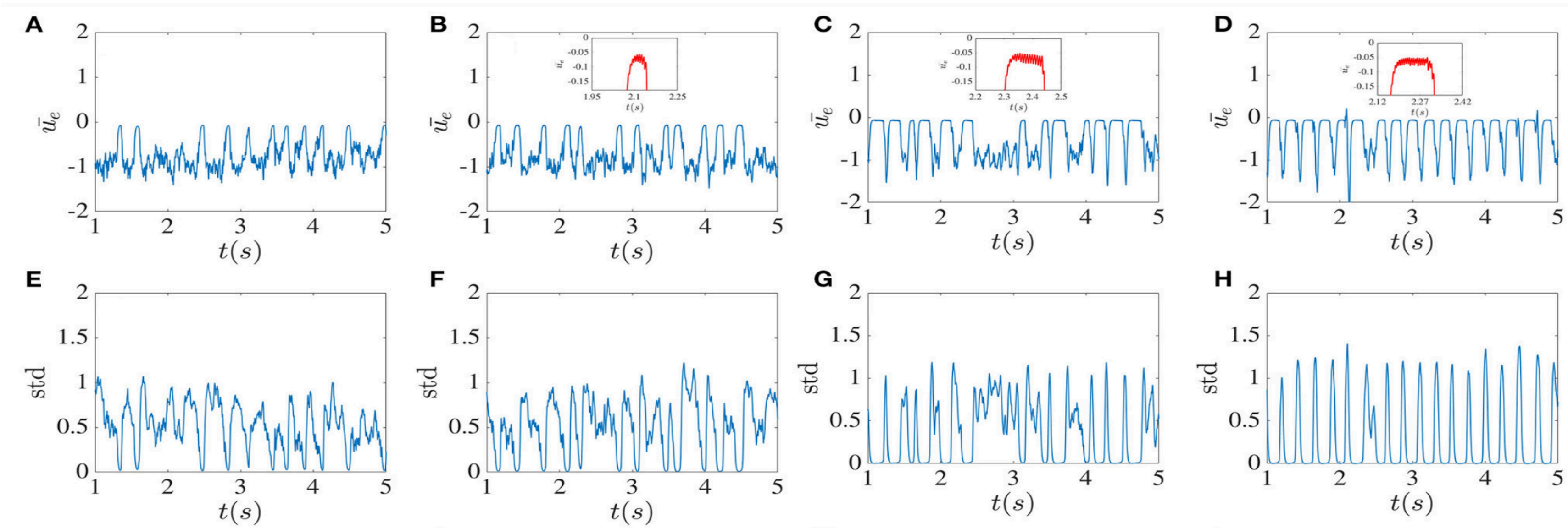
- Recurrent Neural network with global delayed inhibitory global feedback: collapse!

$$\alpha_e^{-1}\frac{du_j}{dt} = -u_j + \frac{1}{n_e}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_e} w_{jk}^{ee}\phi(u_k(t-\tau_l)) + \frac{1}{n_i}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_i} w_{jk}^{ie}\phi(v_k(t-\tau_l)) + \frac{\kappa}{N_i}\sum_{k=1}^{N_i}\phi(v_k(t-T))$$

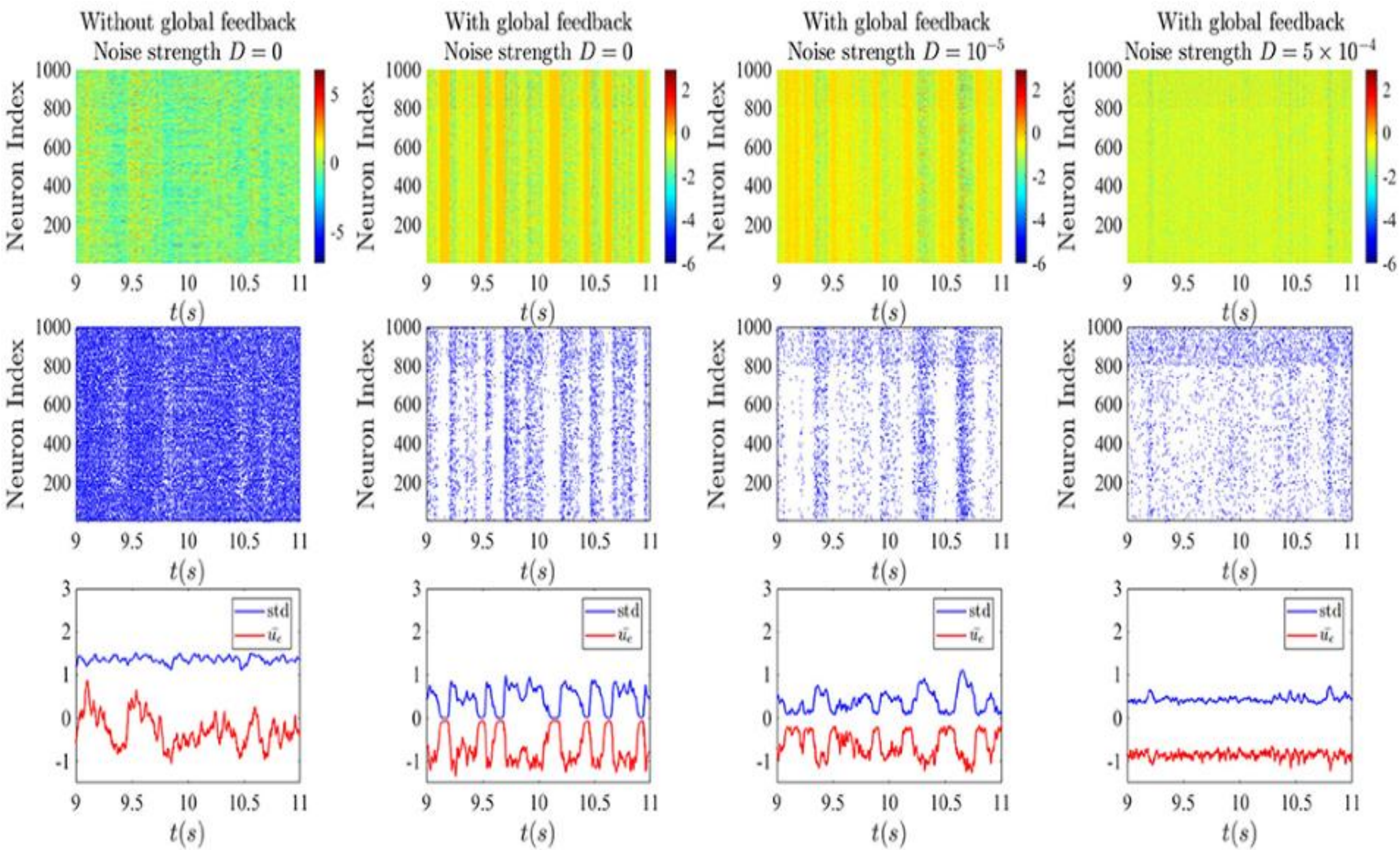
$$\alpha_i^{-1}\frac{dv_j}{dt} = -v_j + \frac{1}{n_e}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_e} w_{jk}^{ei}\phi(u_k(t-\tau_l)) + \frac{1}{n_i}\frac{1}{M}\sum_{l=1}^{M}\sum_{k=1}^{N_i} w_{jk}^{ii}\phi(v_k(t-\tau_l)) + \frac{\kappa}{N_i}\sum_{k=1}^{N_i}\phi(v_k(t-T))$$



Effect of different global time delays *T*=5,10,20 and 30 ms

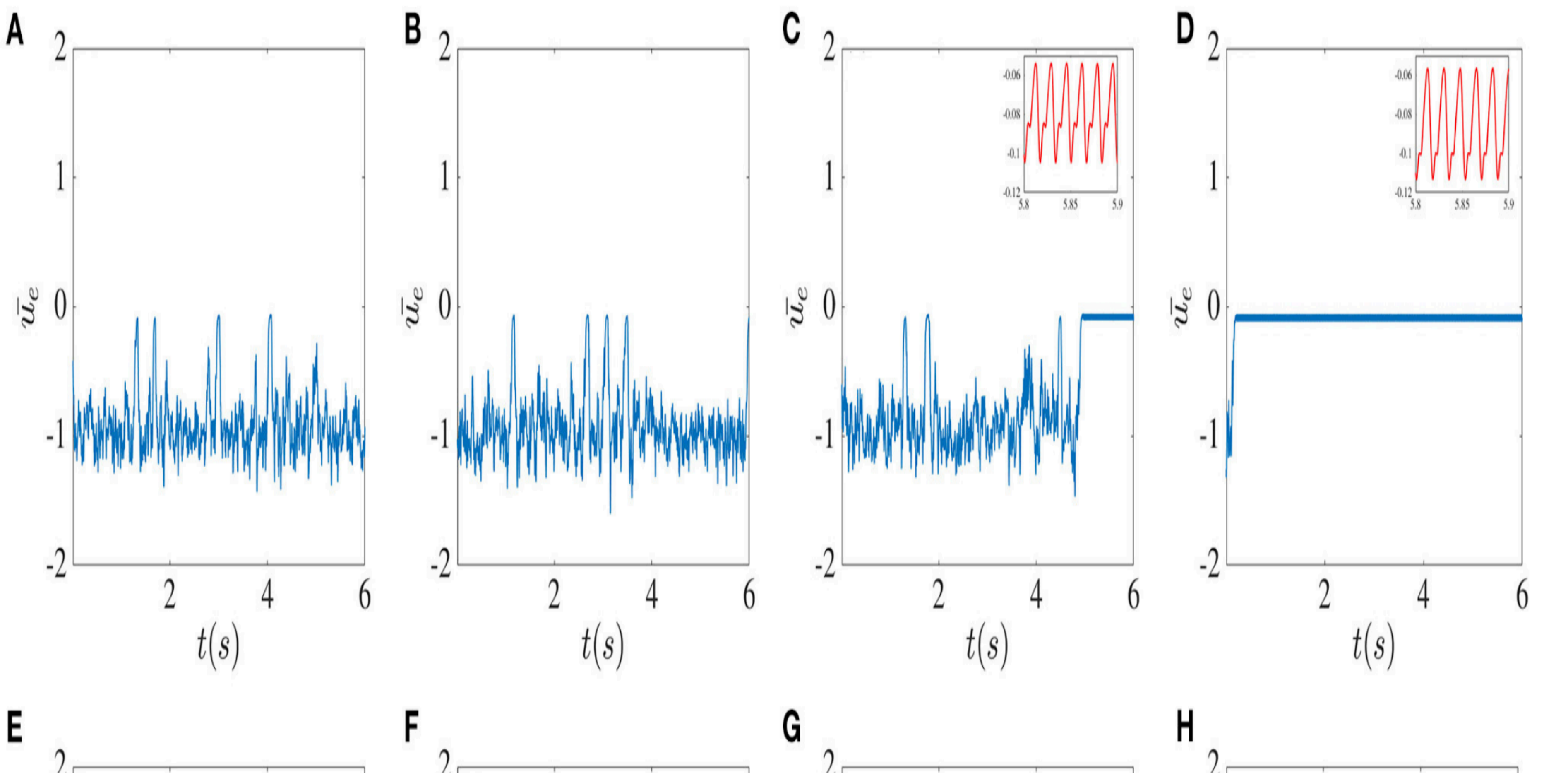


NEURON ACTIVITIES: STRANGE SYNCHED TEMPORAL RANDOMNESS

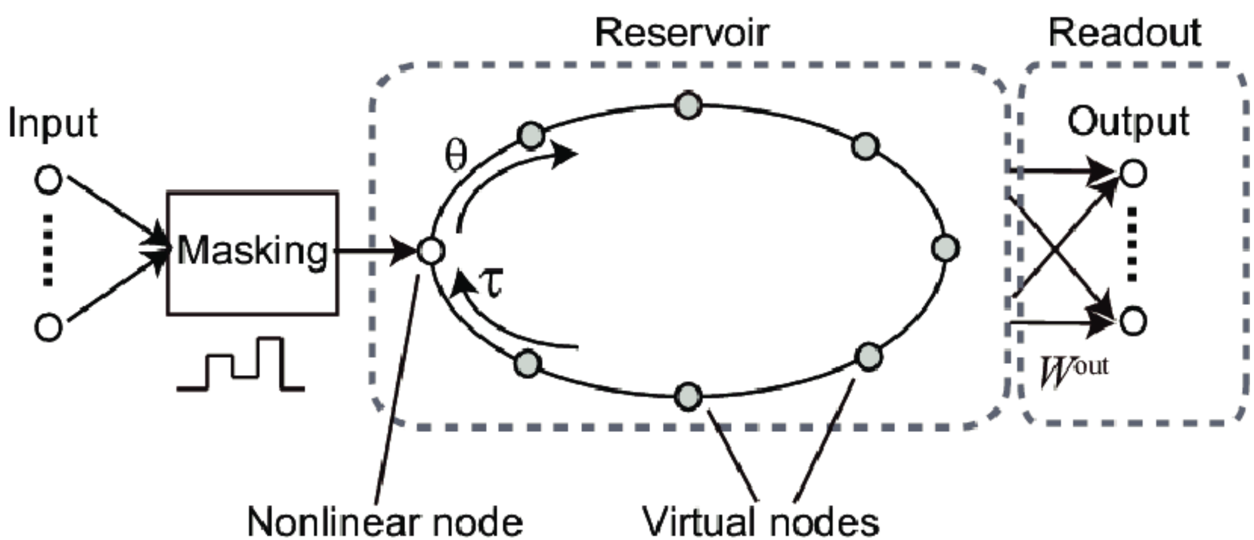


Multiple local delays in the presence of a moderate global delay:
transient chaos → collapse

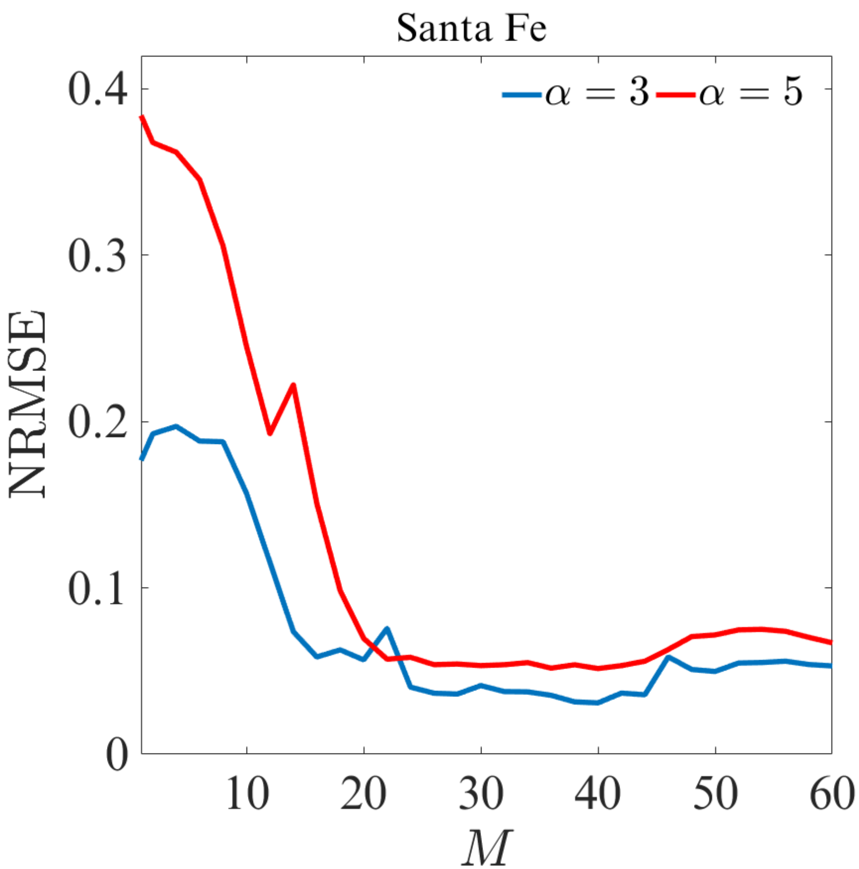
M=6 *M=11* *M=16* *M=21*

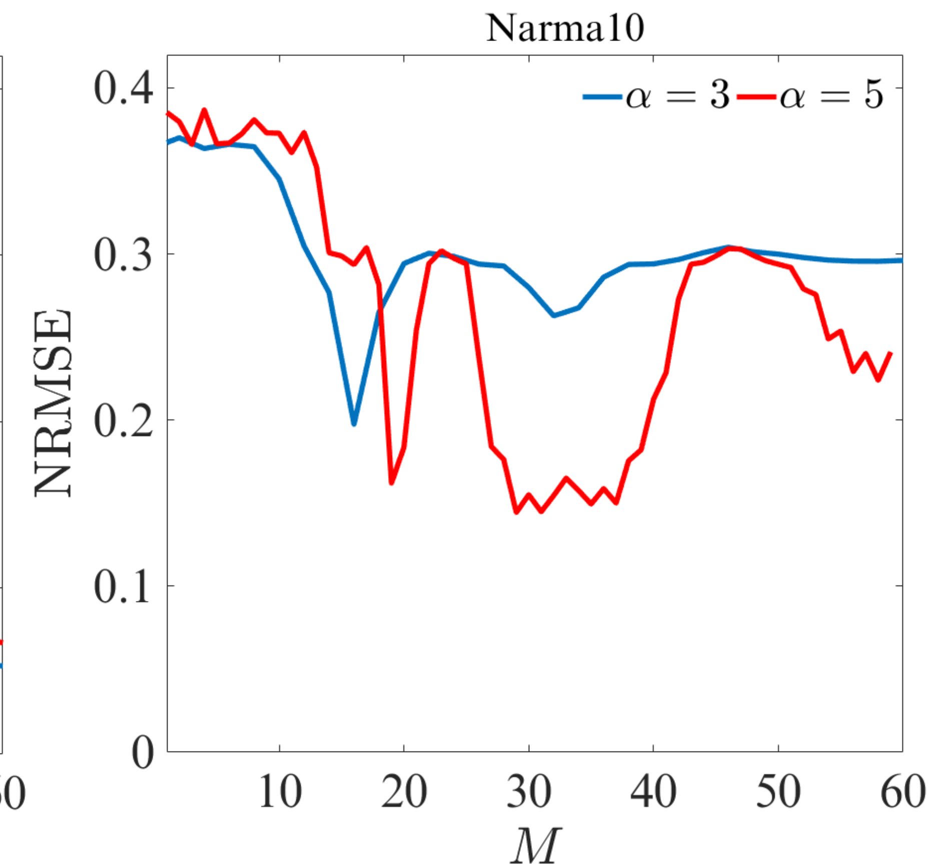


RESERVOIR COMPUTING WITH DELAYED LOOPS
(APPELTANT ET AL. 2011)



RESERVOIR COMPUTING WITH MULTIPLE DELAYS (LASER SYSTEM)





❖SUMMARY

- Transition to simplicity can be abrupt or follow inverse period-doubling sequence in Mackey-Glass blood cell control model

- Large number of delays generally favors simplified dynamics in delayed differential equations in 1-3 variables.

- Chaotic recurrent neural networks may not be simplified by local delays alone.

- The presence of both local delays and global inhibitory feedback may cause collapse to simple dynamics.

- Multiple (but not too many) delays increases entropy: randomness generation

- Multiple delays can benefit reservoir computing:

  → proximity of equilibrium and stronger hyper-chaos

THANK YOU!