Network reconstruction from the observations

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Kuramoto-type network

Coupled oscillators are desctibed via the phase dynamics

$$\dot{\theta}_k = \omega_k + F_k(\theta_1, \ldots, \theta_N), \qquad 1 \leq k, j \leq N$$

Popular examples of different complexity:

$$\text{generic pairwise coupling: } \dot{\theta}_k = \omega_k + \sum_{j=1, j\neq k}^{N} Q_{kj}(\theta_j, \theta_k)$$

$$\text{Winfree-type coupling: } \dot{\theta}_k = \omega_k + S(\theta_k) \sum_{j=1, j\neq k}^{N} A_{kj} g(\theta_j),$$

$$\text{Kuramoto-Daido coupling: } \dot{\theta}_k = \omega_k + \sum_{j=1, j\neq k}^{N} \Gamma_{kj}(\theta_j - \theta_k)$$

$$\text{Hypernetwork: } \dot{\theta}_k = \omega_k + \sum_{j=1, j\neq k}^{N} \sum_{l=1, l\neq k, l>j}^{N} G_{kjl}(\theta_j, \theta_k, \theta_l)$$

Reconstruction approach

We observe the time series $\theta_k(n\Delta t)$ and calculate the time derivatives $\dot{\theta}_1(n\Delta t)$

The coupling function is represented as a Fourier series

$$\Gamma_{1j}(x) = \sum_{m=1}^{M} \left(C_{1j,m} \cos mx + S_{1j,m} \sin mx \right) \ .$$

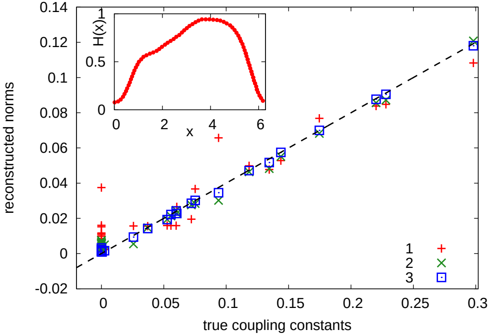
what leads to a set of equations for unknown C, S, ω:

$$\dot{\theta}_1(n) = \omega_1 + \sum_{j=2}^{N} \sum_{m=1}^{M} \big[C_{1j,m} \cos m(\theta_j(n) - \theta_k(n)) + \\ + S_{1jm} \sin m(\theta_j(n) - \theta_k(n)) \big]$$

The unknown parameters are then deterimned by virtue of Singular Value Decomposition through minimisation of squared error.

Example: KD network with realistic coupling function

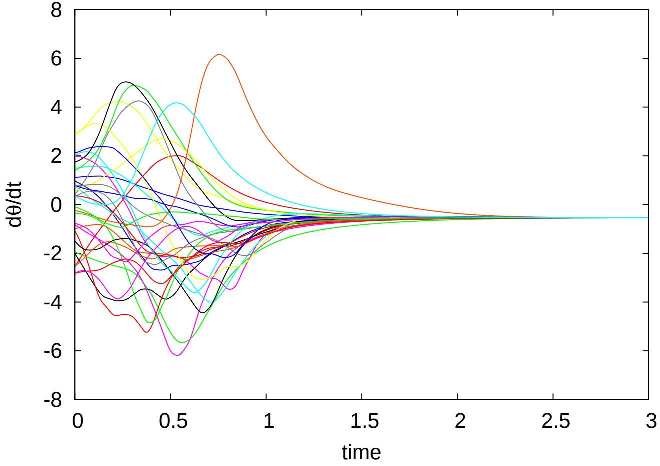
Coupling function from the experimental findings by [Kiss, Zhai and Hudson, PRL (2005)] applied to a random network of 32 oscillators.



Red pluses: reconstruction with $M = 1$, green crosses: $M = 2$, blue squares: $M = 3$ (M is the number of Fourier harmonics used). Dashed line helps to recognize a linear relation between these quantities.

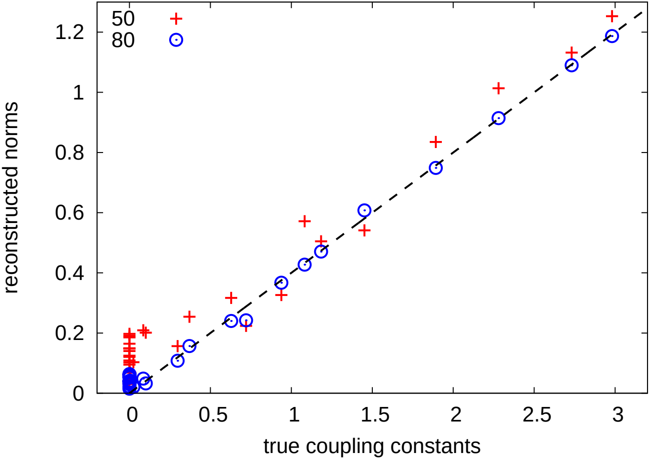
Synchronous dynamics

In the case of synchronous dynamics all the frequencies are identical and direct reconstruction is not possible



Synchronous dynamics: resettings

We apply random resettings of the phases and use transients to reconstruct as above



For large enough number of resettings a good reconstruction is possible

Neural field networks

Following *G. B. Ermentrout and D. H. Terman, Mathematical Foundations of Neuroscience (2010)*, there are two formulations of neural field models:

also state and prove the very general Cohen–Grossberg theorem. Specifically, we are interested in networks of the two general forms:

$$\tau_j \frac{\mathrm{d}u_j}{\mathrm{d}t} + u_j = F_j\left(\sum_k w_{jk} u_k\right), \tag{12.1}$$

$$\tau_j \frac{\mathrm{d}V_j}{\mathrm{d}t} + V_j = \sum_k w_{jk} F_k(V_k). \tag{12.2}$$

The first of these is the so-called 'firing rate' formulation, whereas the second is the voltage formulation. Cowan and Sharp [50] reviewed the history of neural networks

Here F_j are monotonic functions (typically $F(x) \sim 1 + \tanh(x)$)

Reconstruction problem

For the network systems

$$\tau_j \dot{x}_j + x_j = F_j\left(\sum_{k=1}^{n} w_{jk} x_k\right)$$

$$\dot{x}_j = -\gamma_j x_j + \sum_{k=1}^{n} C_{jk} F_k(x_k)$$

under an assumption that time series $x_i(t)\,, i = 1, \ldots, n$ are available, but all the parameters

$$\tau_j, \quad F_j(\cdot), \quad w_{jk}, \qquad\qquad \gamma_j, \quad F_k(\cdot), \quad C_{jk}$$

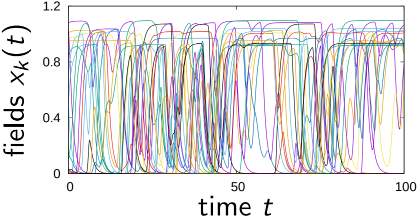
are unknown

we want to reconstruct these unknown parameters from the observations $\vec{x}(t)$

Case I: Firing rate network

$$\tau_j \dot{x}_j + x_j = F_j\left(\sum_{k=1}^{n} w_{jk} x_k\right)$$

We take $n = 100$ and random parameters: $1 - \tau^0 < \tau_j < 1 + \tau^0$, $F_j(u) = \alpha_j/[1 + \exp(-u - \rho_j)]$, $1 - \alpha^0 < \alpha_j < 1 + \alpha^0$
Connection matrix: $w_{jk} = 8 \cdot N(0,1)$ are non-zero with probability $p_c = 0.15$



Dynamics of the fields $\vec{x}(t)$ is chaotic

Idea of reconstruction

Take one of the nodes, e.g. node 1, and define the vector $c_k = w_{1k}$, then

$$\tau_1 \dot{x}_1 + x_1 = F_1 \left(\vec{c} \cdot \vec{x}(t) \right)$$

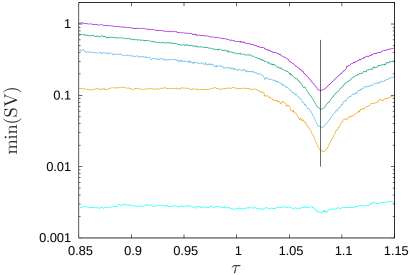
We use monotonicity of F_1:
If $\tau_1 \dot{x}_1(t_1) + x_1(t_1) \approx \tau_1 \dot{x}_1(t_2) + x_1(t_2)$, then $\vec{c} \cdot \vec{x}(t_1) \approx \vec{c} \cdot \vec{x}(t_2)$
This can be re-written as $\vec{c} \cdot (\vec{x}(t_1) - \vec{x}(t_2)) \approx 0$
Now let us collect all the pairs of times t_i, t_m for which $\tau_1 \dot{x}_1(t_i) + x_1(t_i) \approx \tau_1 \dot{x}_1(t_m) + x_1(t_m)$. This yields a large set of relations

$$\vec{c} \cdot \vec{z}(s) = 0, \qquad s = 1, \ldots, M, \quad \text{where} \quad \vec{z} = \vec{x}(t_i) - \vec{x}(t_m)$$

Finding $\vec{c}$ = Singular Value Decomposition problem of finding the null space (vanishing Singular Value) of a $M \times n$ matrix A, composed of M vectors $\vec{z}(s)$ as the rows.

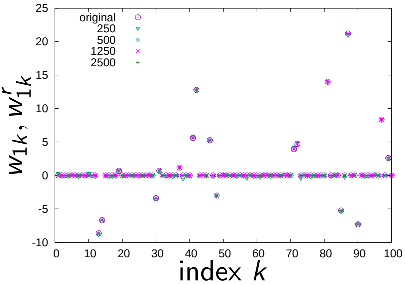
Finding time constant τ_1

Above we assumed that the time constant τ_1 is known – to find this constant we can scan a range of values of τ_1 and to chose that with the minimal Singular Value

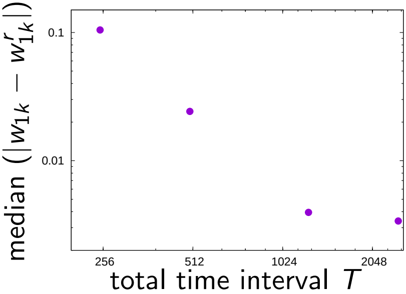


Dependence of the minimal singular value on the parameter τ for different lengths of time series (from top to bottom: total used time intervals $2500, 1250, 500, 250, 100$). The vertical line shows the true value of τ.

Quality of reconstruction

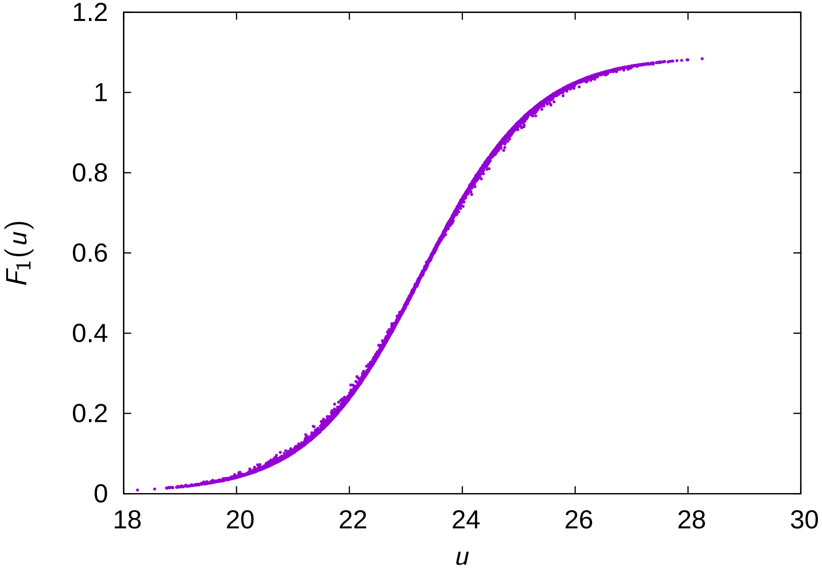


Original coupling constants w_{1k} (circles) and the reconstructed ones w^r_{1k} for the data sets with total used time intervals $T = 2500, 1250, 500, 250$. In these sets the number of data points used for reconstruction was $8961, 4174, 1627, 796$, respectively.

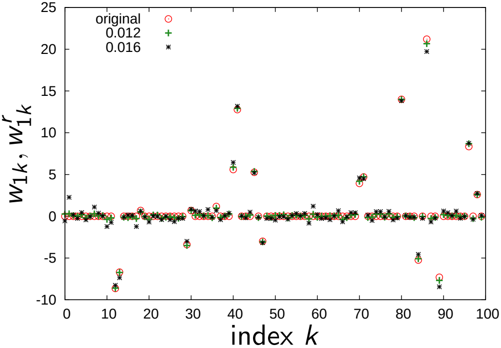


Median errors as functions of the total time interval used.

Reconstruction of the gain function



Noise sensitivity



Original coupling constants w_{1j} (circles) and the reconstructed ones w^r_{1j} for the data sets with total used time interval $T = 5000$, for two values of standard deviation of observational Gaussian noise : $\delta = 0.012$ and $\delta = 0.016$.

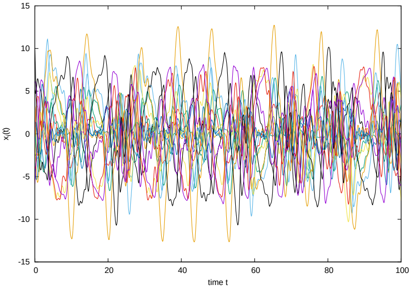
The noisy data sets have been pre-processed with the Savitzky-Golay filter of order (16,16,4), the same filter has been used to calculate the derivatives.

Case II: Voltage network

$$\dot{x}_j = -\gamma_j x_j + \sum_{k=1}^{n} C_{jk} F_k(x_k)$$

Sompolinsky et al. [Phys. Rev. Lett. v. 61 n. 3 p. 259] showed that this neural activity model with a sigmoidal gain function $F(x) = \tanh(x)$ demonstrates chaos for strong enough random coupling.

We use random uniformly distributed $1 - \Delta < \gamma_j < 1 + \Delta$ and C_{jk} from a Gaussian distribution. Here $n = 16$.



Matrix inversion

$$\dot{x}_j = -\gamma_j x_j + \sum_{k=1}^{n} C_{jk} F_k(x_k)$$

Because functions $F_k()$ are not known, we cannot use the procedure from the firing-rate network setup above, but assuming that the matrix C is invertable $W = C^{-1}$ we can rewrite the system of equations as

$$F(x_j) = \sum W_{ji}(\dot{x}_i + \gamma_i x_i)$$

Reconstruction idea

$$F(x_j) = \sum W_{ji}(\dot{x}_i + \gamma_i x_i)$$

For any function F (not necesserely monotonic one) if $x_j(t_1) \approx x_j(t_2)$, then

$$W_{ji}\left[\dot{x}_i(t_1) + \gamma_i x_i(t_1) - \dot{x}_i(t_2) - \gamma_i(t_2)\right] \approx 0$$

Collecting all time instants t_l, t_m for which $x_j(t_l) \approx x_j(t_m)$, we obtain a large set of equations

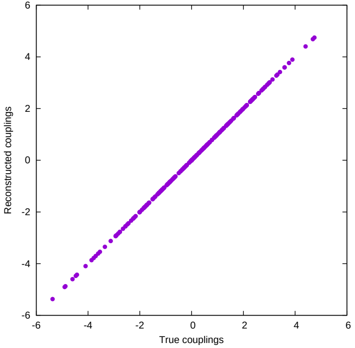
$$W_{ij}\, z_j(s) = 0, \quad s = 1, \ldots, M,$$
$$z_j(s) = \left[\dot{x}_i(t_l) + \gamma_i x_i(t_l) - \dot{x}_i(t_m) - \gamma_i(t_m)\right]$$

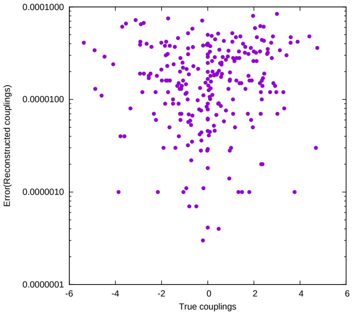
from which the matrix W can be found via Singular Value Decomposition as Null-Space-Vectors.

Known time constants

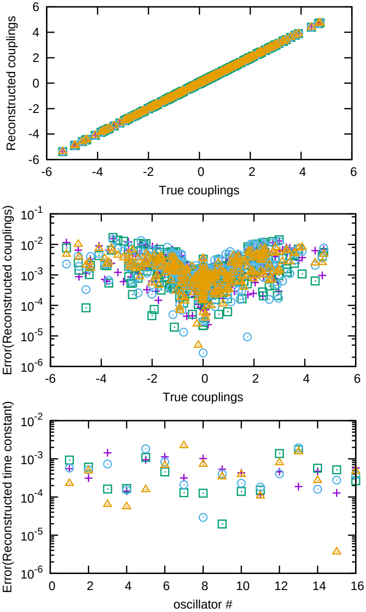
$$F(x_j) = \sum W_{ji}(\dot{x}_i + \gamma_i x_i)$$

For the reconstruction we need to possess all the time constants γ_i. If they are known, the task is easy.





Unknown time constants



Here we have to find all the time constants γ_i – this was accomplished via Simulating Annealing serach for the minimal Singular Value.

Reconstructed coupling coefficients and time constants. Four symbols show four independent runs of the simulated annealing routine. Number of points used 10000.

Conclusions

- Oscillatory networks in phase dynamics representation can be reconstructed for different complexity levels (Kuramoto-Daido coupling, WInfree coupling, hypernetwork)
- In case of full or partial synchrony, phase resettings help
- For neural networks, only general structural knowledge about the systems is needed, but long and possibly noise-free time series is required
- Scalar local dynamics
- Similar approach to networks with time-delay coupling: Sysoev, Ponomarenko, et. al, Phys. Rev. E **89** 062911 (2014); **94** 052207 (2016)

Publications: Phys. Rev. E **93** 062313 (2016), EPL, v. 119, 30004 (2017), Phys. Lett. A, v. 382, 147-152 (2018)