Common noise vs Coupling in Oscillator Ensembles

A. Pikovsky

Institut for Physics and Astronomy, University of Potsdam, Germany

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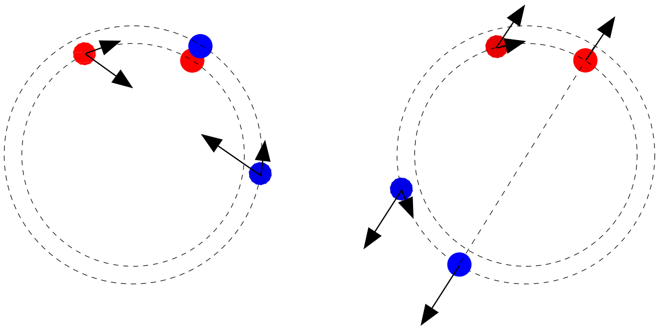
Collaborators: A. Dovlatova, D. Goldobin, M. Rosenblum

Two coupled deterministic oscillators

Interaction of two periodic oscillators may be attractive ore repulsive: one observes **in phase** or **out of phase** synchronization, correspondingly



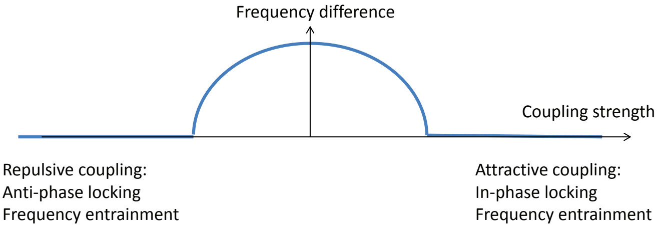
Two coupled deterministic oscillators: Adler equation

$$\dot{\varphi}_1 = \omega + \delta\omega + \mu \sin(\varphi_2 - \varphi_1) \qquad \dot{\varphi}_2 = \omega - \delta\omega + \mu \sin(\varphi_1 - \varphi_2)$$

Adler equation for the phase difference $\theta = \varphi_1 - \varphi_2$:

$$\dot{\theta} = 2\delta\omega - 2\mu \sin\theta$$

Frequency difference $\langle\dot{\theta}\rangle$ vs coupling strength μ for fixed mismatch $\delta\omega$:



Phase locking vs Frequency entraiment

Phase locking:

$$\varphi_1(t) - \varphi_2(t) \approx \text{const}$$

Frequency entrainment:

$$\nu_1 = \langle \dot{\varphi}_1 \rangle = \nu_2 = \langle \dot{\varphi}_1 \rangle$$

Many coupled deterministic oscillators: Kuramoto model

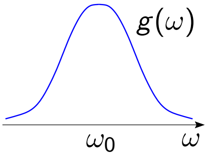
Describes an ensemble of phase oscillators with all-to-all coupling

$$\dot{\phi}_i = \omega_i + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\phi_j - \phi_i)$$

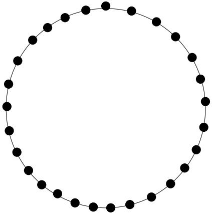
Can be written as a mean-field coupling

$$\dot{\phi}_i = \omega_i + \mu(-X \sin\phi_i + Y \cos\phi_i) \qquad X + iY = Z = Re^{i\Phi} = \frac{1}{N} \sum_j e^{i\phi_j}$$

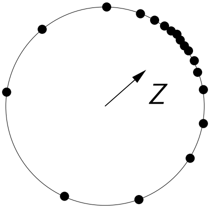
The natural frequencies are distributed around some mean frequency ω_0



Synchronisation transition



negative (repulsive) and small μ: no synchronization, phases are distributed uniformly, mean field $= 0$



large positive μ: synchronization, distribution of phases is non-uniform, mean field $\neq 0$

Phase locking vs Frequency entraiment

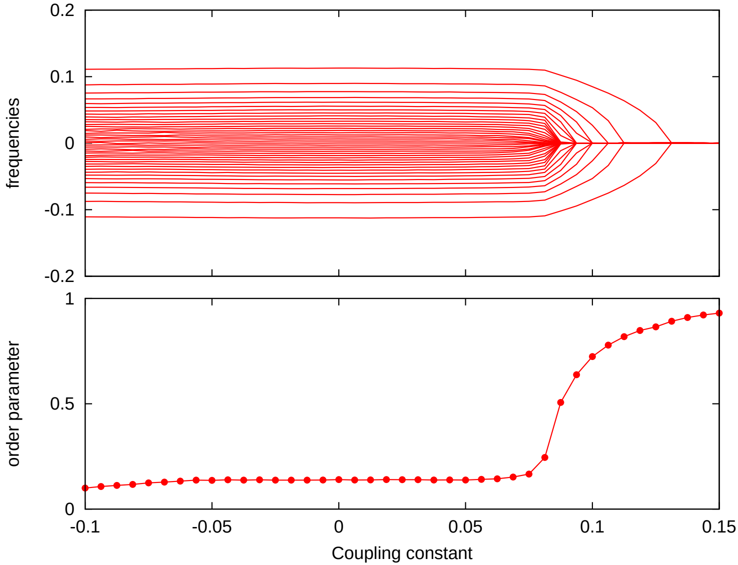
Phase locking is characterized by the mean field

$$R = \frac{1}{N}\left|\sum_{1}^{N} e^{i\varphi_k}\right|$$

Beyond the synchronization threshold $R > 0$ indicating that the phases are close to each other

Frequency entrainment: A cluster of oscillators with exactly equal frequencies appears, all other frequencies are pulled together

Ensemble of coupled oscillators



Synchronization by common noise

Two or more identical oscillators driven by the same noise:

$$\dot{\varphi}_k = \omega + \xi(t) \sin \varphi_k$$

Equation for the (small) phase difference:

$$\frac{\frac{d}{dt}\delta\varphi}{\delta\varphi} = \frac{d}{dt} \ln \delta\varphi = \xi(t) \cos \varphi$$

Averaging yields the Lyapunov exponent:

$$\lambda = \langle \frac{d}{dt} \ln \delta\varphi \rangle = \langle \xi(t) \cos \varphi \rangle < 0$$

Negative Lyapunov exponent: the fully synchronous state

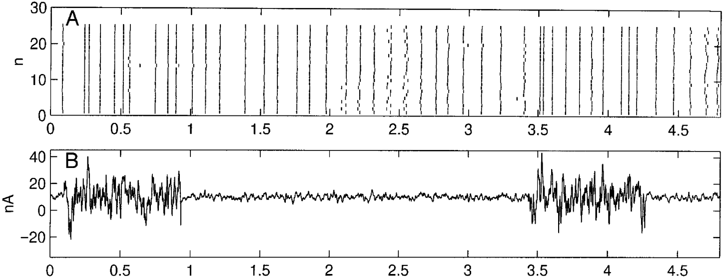
$$\varphi_1 = \varphi_2 = \ldots = \varphi_N$$

is stable
With a small frequency mismatch the phases are close to each other $\varphi_1 \approx \varphi_2$ but do not coincide

Reliability of neuron spikes (Mainen and Sejnowski, 1995)

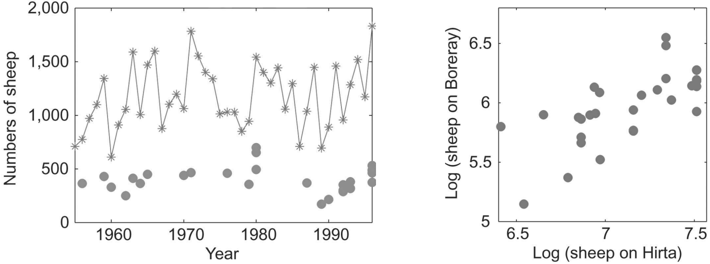
A neuron is subject to the same noisy forcing ⇒ the same response (after *Hunter et al, J. Neurophysiol., 1427 (1998)*)

FIG. 4. Spike time reliability in *Aplysia* motoneuron with aperiodic inputs. Superposed voltage traces from 10 different trials recorded from a buccal motoneuron for 4 different input signals. *A*: broadband aperiodic input



Common noise in ecology: Moran effect

P. A. P. Moran (Aust. J. Zool. **1**, 291, 1953) mentioned that two linear systems driven by correlated noises produce correlated outputs



Temporal dynamics of feral sheep populations on the St. Kilda archipelago (Grenfell et al, Nature, **394**, 674, 1998)

Phase locking vs Frequency entrainment

Ensemble of uncoupled oscillators with a distribution of frequencies under common noise

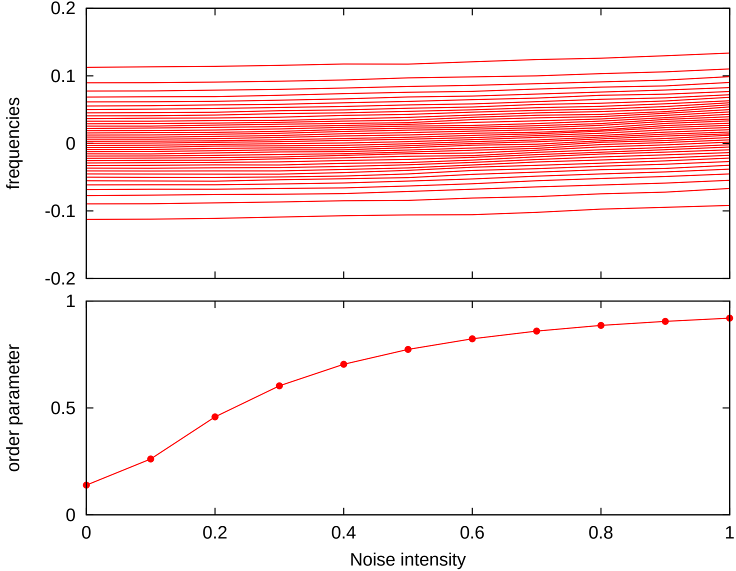
Phase locking: Mean field

$$R = \frac{1}{N}\left|\sum_{1}^{N} e^{i\varphi_k}\right|$$

is large

Frequency entrainment:
Frequencies are not affected by the common noise – they remain the (nearly) natural ones

Ensemble of uncoupled oscillators under common noise

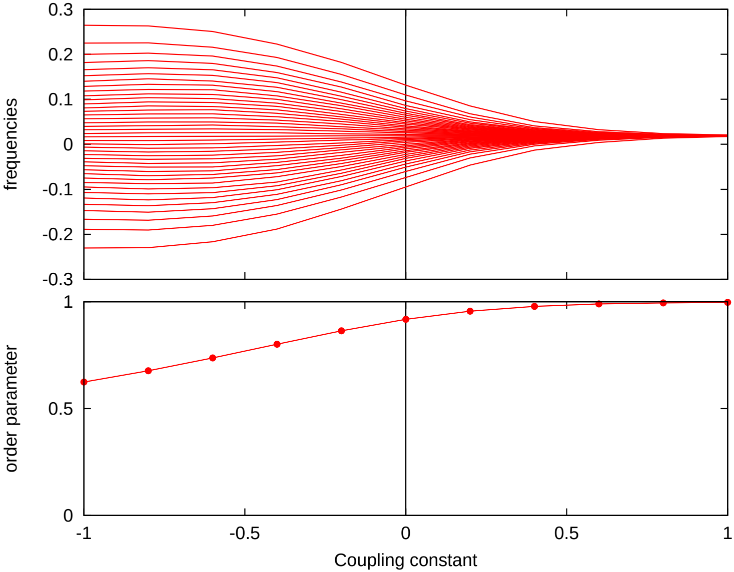


Common noise + coupling

Common noise + attractive coupling:
Both factors lead to a large order parameter
Frequencies are pulled together due to attractive coupling
Phase locking and Frequency entrainment

Common noise + repulsive coupling:
Noise leads to a large order parameter (at least if the coupling is not too large)
Frequencies are dispersed due to repulsive coupling
Phase locking and Frequency anti-entrainment

Ensemble with common noise and coupling



Model: ensemble of phase oscillators with common noise and Kuramoto-type coupling

We consider thermodynamic limit $N \to \infty$ with a Lorentzian distribution of natural frequencies

$$g(\Omega) = \frac{\gamma}{\pi[\gamma^2 + (\Omega - \Omega_0)^2]}$$

Langevin equations:

$$\dot{\varphi}_\Omega = \Omega + \sigma\xi(t)\sin\varphi_\Omega + \mu R \sin(\Phi - \varphi_\Omega)\ ,$$
$$\langle \xi(t)\xi(t')\rangle = 2\delta(t - t')\ .$$

Here the mean field is defined as

$$Z = Re^{i\Phi} = \langle e^{i\varphi}\rangle = \int_{-\infty}^{\infty} d\Omega\, g(\Omega) \int_0^{2\pi} d\varphi_\Omega\, e^{i\varphi_\Omega} w(\varphi_\Omega, t)\ ,$$

Ott-Antonsen formulation

Under the assumption of a particular parametrization of the probability density, the order parameter Z obeys a stochastic differential equation

$$\dot{Z} = i\Omega_0 Z - \gamma Z + \frac{\mu Z(1 - |Z|^2) - \sigma(1 - Z^2)\xi(t)}{2}$$

It contains four parameters:

- the basic frequency Ω_0 (which, in contradistinction to the usual Kuramoto model, cannot be simply shifted to zero, because the noise term breaks the frequency-shift invariance)
- the noise intensity σ^2
- the coupling constant μ
- the width of the distribution of natural frequencies γ.

After averaging over fast frequency Ω_0

Assuming that Ω_0 is large, we average over fast oscillations and obtain for an order parameter $J = R^2/(1 - R^2)$ the following stochastic equation

$$\frac{dJ}{dt} = \mu J - 2\gamma J(1 + J) + \frac{\sigma^2}{2}(J + 1/2) - \sigma\sqrt{\frac{(1 + J)J}{2}}\zeta_1(t)$$

with new effective noise $\zeta_1(t)$
Relation $R \Leftrightarrow J$:

$$R = 0 \quad \Leftrightarrow \quad J = 0 \qquad\qquad R = 1 \quad \Leftrightarrow \quad J = \infty$$

Identical oscillators

Here $\gamma = 0$ and

$$\frac{dJ}{dt} = \mu J + \frac{\sigma^2}{2}(J + 1/2) - \sigma\sqrt{\frac{(1+J)J}{2}}\zeta_1(t)$$

The limit $J \to \infty$ (full synchrony) is simple, here

$$\frac{1}{J}\frac{d}{dt}J = \frac{d}{dt}\ln J = \mu + \frac{\sigma^2}{2} + \frac{\sigma}{\sqrt{2}}\zeta_1(t)$$

Quantity $\lambda = -\mu - \frac{\sigma^2}{2}$ is the Lyapunov exponent determining stability of the full synchrony

$\mu > -\sigma^2/2$: full synchrony stable
$\mu < -\sigma^2/2$: full synchrony unstable

“Bistability”

Here $\gamma = 0$ and

$$\frac{dJ}{dt} = \mu J + \frac{\sigma^2}{2}(J + 1/2) - \sigma\sqrt{\frac{(1+J)J}{2}}\zeta_1(t)$$

For $-\sigma^2/2 < \mu <$ there is a “bistable” situation: full synchrony is stable, and the coupling is repulsing so that the asynchronous state $J = 0$ is stable in absence of noise

asynchronous state $J = 0$: noise is additive $\rightarrow$ broad distribution of J

synchronous state $J = \infty$: noise is multiplicative, no fluctuations around this state

synchronous state always wins and is an absorbing one

Nonidentical oscillators

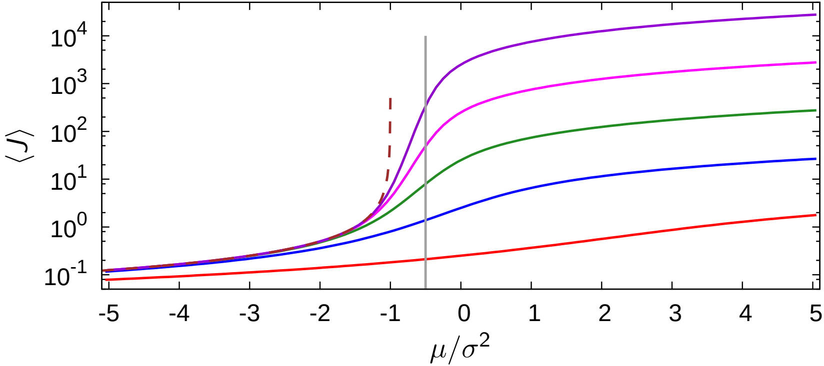
Distribution of J can be found analytically:

$$W(J;\gamma,\mu,\sigma^2) = \frac{(1+J)^{2\mu\sigma^{-2}}\exp[-4\gamma\sigma^{-2}(1+J)]}{(4\gamma\sigma^{-2})^{1+2\mu\sigma^{-2}}\Gamma(2\mu\sigma^{-2}+1,4\gamma\sigma^{-2})}$$

where $\Gamma(m,x)$ is the upper incomplete Gamma function. The average values of the order parameters are

$$\langle R^2\rangle = 1 - \frac{4\gamma}{\sigma^2}\frac{\Gamma\left(\frac{2\mu_\beta}{\sigma^2},\frac{4\gamma}{\sigma^2}\right)}{\Gamma\left(\frac{2\mu_\beta}{\sigma^2}+1,\frac{4\gamma}{\sigma^2}\right)}, \quad \langle J\rangle = \frac{\sigma^2}{4\gamma}\frac{\Gamma\left(\frac{2\mu_\beta}{\sigma^2}+2,\frac{4\gamma}{\sigma^2}\right)}{\Gamma\left(\frac{2\mu_\beta}{\sigma^2}+1,\frac{4\gamma}{\sigma^2}\right)} - 1$$

Nonidentical oscillators



Values of $\langle J \rangle$ for different γ/σ^2 as functions of μ/σ^2. From top to bottom: $\gamma/\sigma^2 = 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1$. Brown dashed line corresponds to the system of identical oscillators $\gamma = 0$. Vertical grey line shows the border of stability of the fully synchronous state for $\gamma = 0$

Nonidentical oscillators: frequencies

For individual phases (differences from the mean field phase $\theta_\omega = \varphi_\Omega - \Phi$) we have stochastic equations

$$\dot{J} = \mu J - 2\gamma J(1+J) + \frac{\sigma^2}{2}(J+1/2) - \sigma\sqrt{\frac{(1+J)J}{2}}\zeta_1(t)$$

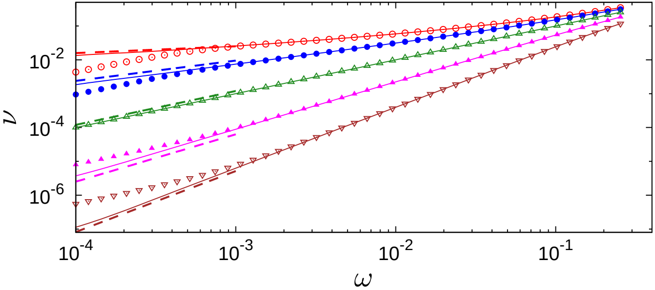
$$\dot{\theta} = \omega - \mu\sqrt{\frac{J}{1+J}}\sin\theta - \frac{\sigma^2}{4}\frac{(J+1/2)}{\sqrt{J(1+J)}}\sin\theta$$

$$+\frac{\sigma}{\sqrt{2}}\sin\theta\zeta_1(t) + \frac{\sigma}{\sqrt{2}}\left(\cos\theta - \frac{(J+1/2)}{\sqrt{J(1+J)}}\right)\zeta_2(t)$$

Here $\omega = \Omega - \Omega_0$ is the mismatch to the mean frequency

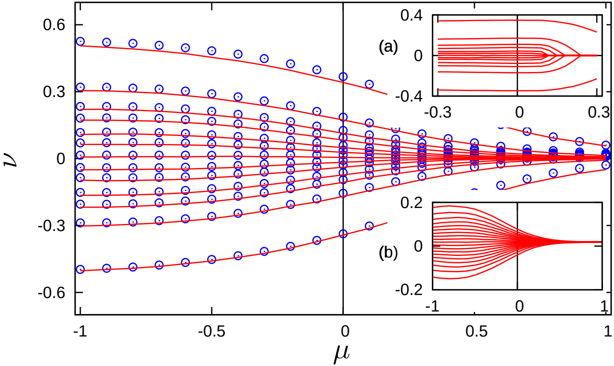
Solutions of an approximate equation for frequencies

If we assume $J = const$, for the distribution of θ we get a closed Fokker-Planck equation, which can be solved numerically



Observed frequencies $\nu = \langle \dot{\theta} \rangle$ vs natural frequencies ω. Solid lines: solutions for $J = \infty$, markers: solutions for $\langle J \rangle = 10$. From top to bottom: $\mu/\sigma^2 = -0.4,\ -0.2,\ 0,\ 0.2,\ 0.4$. Dashed lines have slopes $1 + 2\mu/\sigma^2$.

Illustration of frequency dispersion



Observed frequencies ν vs coupling strength μ

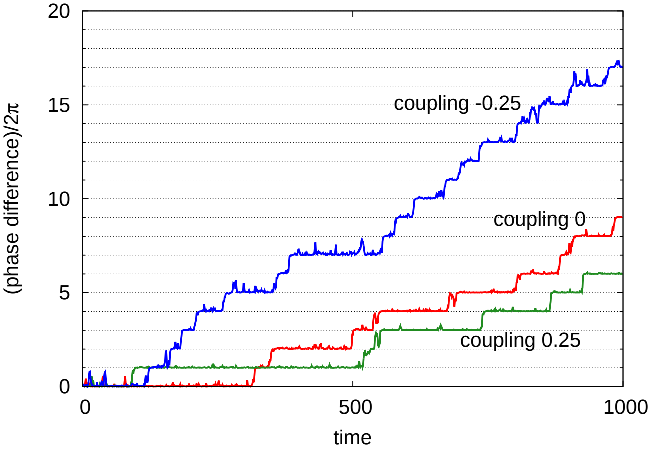
Markers: direct simulations of the population of 21 phase oscillators (for better visibility, not all frequencies are depicted)

Solid lines: simulations of the Ott-Antonsen equations, valid in the thermodynamic limit, for the same individual frequencies.

The inset (a) shows the case without noise $\sigma = 0$.

The inset (b) shows the case of a Gaussian distribution of frequencies.

Illustration of the phase dynamics



In all cases the phase difference for two oscillators is predominantly zero (mod 2π): phase locking

Attractive coupling: phase slips less frequent – frequency entrainment

Repulsive coupling: phase slips more frequent – frequency repulsion

Identical oscillators: Is clustering possible?

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Common noise induces clustering in populations of globally coupled oscillators

S. Gil[1(a)], Y. Kuramoto[2] and A. S. Mikhailov[1]

Existence of clusters is excluded by the Watanabe-Strogatz theory of integrability of identical oscillators (if the noise is interpreted in Stratonovich sense): Clustering is an artifact of numerical unaccuracy

Conclusions

- Nearly full stochastic description of ensembles of coupled oscillators under common noise in the Ott-Antonsen regime is possible
- For identical oscillators: asymmetric bistability in presence of noise and repulsive coupling, where the full synchrony always wins
- For identical oscillators: clustering for a Kuramoto-type coupling not possible, but can observed due to numerical inaccuracy
- For nonidentical oscillators: **Phase locking and frequency anti-entrainment for repulsive coupling**

See Sci. Reports **6**, 38158 (2016), EPJ-ST **226** 1921 (2017), PRE **96** 062204 (2017), Chaos **29** 033127 (2019)