Quantifying Stability Complex Networks and its Application to Neuroscience

Jürgen Kurths

Potsdam Institute for Climate Impact Research
&
Institute of Physics, Humboldt-Universität zu Berlin

POTSDAM INSTITUTE FOR CLIMATE IMPACT RESEARCH
P I K

juergen.kurths@pik-potsdam.de

http://www.pik-potsdam.de/members/kurths/

Main Collaborators:

- **Sabine Auer, Jobst Heitzig, Frank Hellmann, Peter Menck, Paul Schultz** PIK Potsdam & Humboldt Uni Berlin
- **Jinqiao Duan, Meng Zhan, Yayun Zheng** HUST, Wuhan
- **Siyang Leng, Wei Lin, Peng Ji** Fudan Uni
- **Piotr Brzeski, Tomasz Kapitaniak, Przem Perlikowski,** Lodz Univ Technol, Poland
- **Vladimir Klinchuk,** IAP, Nizhny Novgorod

Contents

Complex Networks

Network of Networks

Interconnected Networks

Interdependent Networks

Multiplex Networks

Multilayer Networks…

Power Grids

Intended Solution:

stable synchronized behaviour along the whole network of networks

How to control such networks?

Pinning Control (which nodes?)

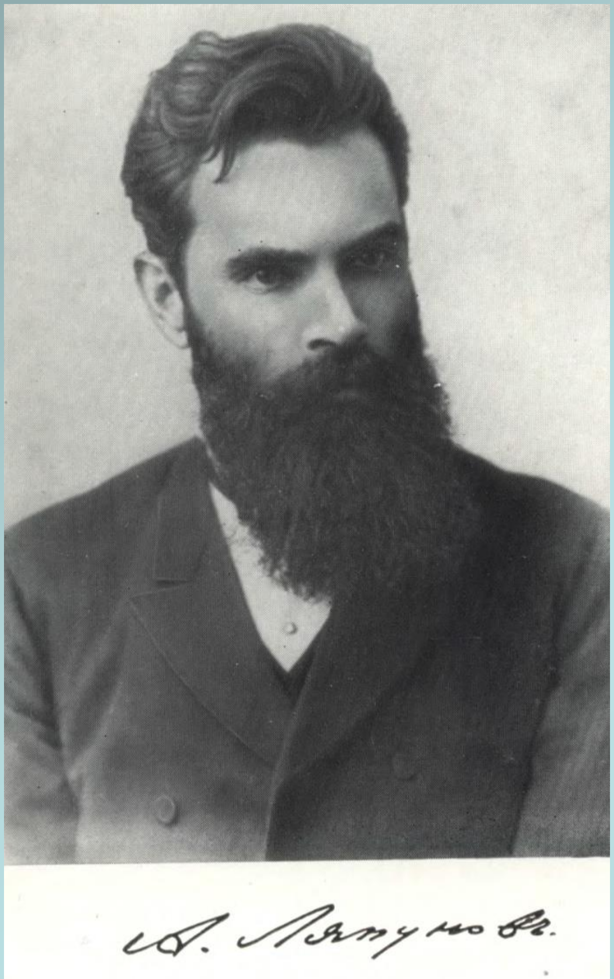
Highly Non-trivial Task

Monster blackouts

Failing of Control!!!

Stability

Stability of Dynamical Systems



Alexandr Mikhailovich Lyapunov (1857 – 1918)

- Student of P. L. Chebyshev and friend of A. A. Markov
- Master: On the stability of ellipsoidal forms of equilibrium of rotating fluids (1884) – french translation (1904)
- PhD: The general problem of the stability of motion (1892)
- 1893 – full Prof. Kharkiv Univ
- 1902 – St. Petersburg (followed Chebyshev)

Alexandr Mikhailovich Lyapunov

- Lyapunov was the first to consider modifications necessary in *nonlinear systems* to the linear theory of stability based on linearizing near a point of equilibrium
- The equilibrium x_ε of the system is said to be *Lyapunov stable*, if for every ($\forall\ \varepsilon > 0$) and ($\forall\ t_0$), there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that,

  if $|x(t_0)-x_\varepsilon| < \delta$, then $|x(t)-x_\varepsilon| < \varepsilon$, for every $t \geq 0$.
- Extension to asymptotical and exponential stability

Stability of Complex Networks

(Synchronized Dynamics)

Weighted Network of N Identical Oscillators

$$\begin{aligned}\dot{\mathbf{x}}_i &= \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} W_{ij} A_{ij} [\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)], \\ &= \mathbf{F}(\mathbf{x}_i) - \sigma \sum_{j=1}^{N} G_{ij} \mathbf{H}(\mathbf{x}_j), \quad i = 1, \ldots, N,\end{aligned}$$

F – dynamics of each oscillator

H – output function

G – coupling matrix combining adjacency A and weight W

$G_{ij} = -W_{ij}$ for $i \neq j$ $\qquad$ $G_{ii} = \sum_j W_{ij} A_{ij} = S_i$

S_i - intensity of node i (includes topology and weights)

General Condition for Synchronizability

Stability of synchronized state

$$\{\mathbf{x}_i = \mathbf{s}, \forall\, i \,|\dot{\mathbf{s}} = \mathbf{F}(\mathbf{s})\}$$

N eigenmodes of

$$\dot{\xi}_i = [D\mathbf{F}(\mathbf{s}) - \sigma\lambda_i D\mathbf{H}(\mathbf{s})]\xi_i,$$

λ_i *i*th eigenvalue of G

Main results

Synchronizability universally determined by:

- mean degree K and

- heterogeneity of the intensities

$$\frac{S_{\max}}{S_{\min}} \quad \text{or} \quad \frac{\Omega}{S_{\min}}$$

$S_{\min}$, $S_{\max}$ - minimum/ maximum intensities

(Motter, Zhou, Kurths, PRL 2006)

IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART B: CYBERNETICS, VOL. 40, NO. 3, JUNE 2010 881

Second-Order Consensus for Multiagent Systems With Directed Topologies and Nonlinear Dynamics

Wenwu Yu, *Student Member, IEEE*, Guanrong Chen, *Fellow, IEEE*, Ming Cao, *Member, IEEE*, and Jürgen Kurths

358 IEEE TRANSACTIONS ON CYBERNETICS, VOL. 43, NO. 1, FEBRUARY 2013

Distributed Synchronization in Networks of Agent Systems With Nonlinearities and Random Switchings

Yang Tang, *Member, IEEE*, Huijun Gao, *Senior Member, IEEE*, Wei Zou, and Jürgen Kurths

IEEE TRANSACTIONS ON INDUSTRIAL ELECTRONICS, VOL. 63, NO. 2, FEBRUARY 2016 1299

Tracking Control of Networked Multi-Agent Systems Under New Characterizations of Impulses and Its Applications in Robotic Systems

Yang Tang, *Member, IEEE*, Xing Xing, Hamid Reza Karimi, *Senior Member, IEEE*, Ljupco Kocarev, *Fellow, IEEE*, and Jürgen Kurths

Stability of Networks

Synchronizability –
Master Stability Formalism

Pecora&Carrol (1998) –

based on **local stability**

Synchronizability – Master Stability Formalism
(Pecora&Carrol (1998)

Synchronizability Ratio $R = \lambda_{\max}/\lambda_{\min}$

Stability Interval for coupling strength K

$$K \in I_s = (\alpha_1/\lambda_{\min}, \alpha_2/\lambda_{\max}),$$

Synchronizability condition $R < \alpha_2/\alpha_1$

Stability/synchronizability in small-world (SW) networks

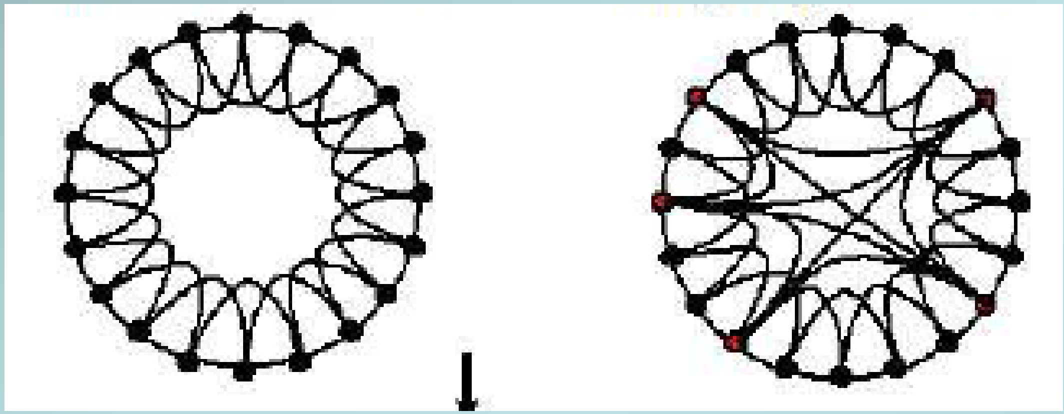
Small-world (SW) networks

(Watts, Strogatz, 1998 – WS-networks)

F. Karinthy hungarian writer –

SW hypothesis (1929)

Small-world Networks



k nearest neighbour connections

Nearest neighbour and a few long-range connections

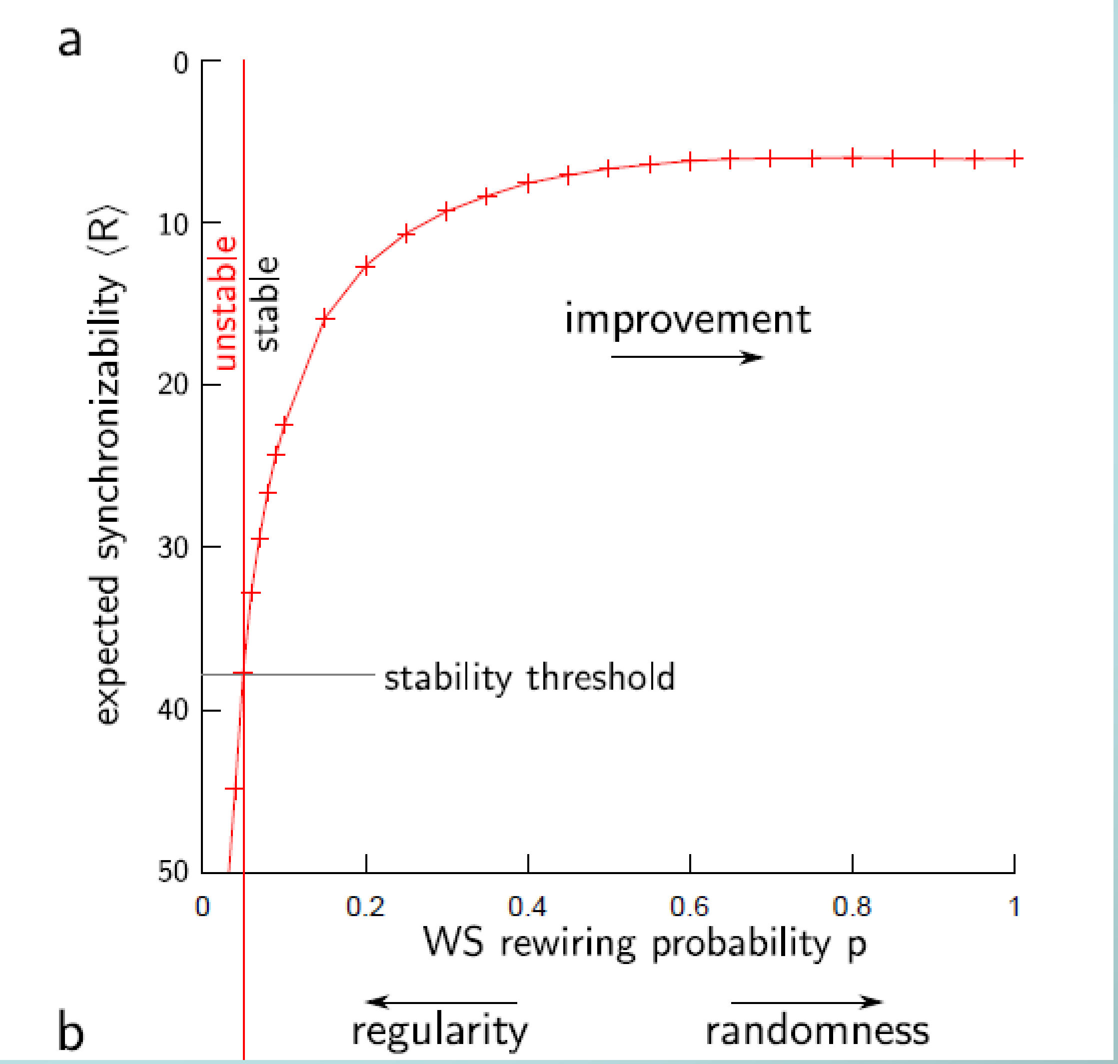
Regular ➔ Complex Topology

$$\dot{\mathbf{r}}_i = \mathbf{F}(\mathbf{r}_i) + K\sum_j A_{ij}[\mathbf{H}(\mathbf{r}_j) - \mathbf{H}(\mathbf{r}_i)] = \mathbf{F}(\mathbf{r}_i) - K\sum_j L_{ij}\mathbf{H}(\mathbf{r}_j),$$

$$\begin{aligned}
\dot{x}_i &= -y_i - z_i - K\sum_{i=1}^{N} L_{ij}x_j \\
\dot{y}_i &= x_i + ay_i \\
\dot{z}_i &= b + z_i(x_i - c)
\end{aligned}$$

Chosen: a = b = 0.2, c = 7.0 ➔ R < 37.88

Chaotic Rössler oscillators, N = 100



Main Result: SW-Network **best synchronizable for most random** SW-networks

Puzzle!

MSF – local stability (Lyapunov stability)

How to go beyond (not only small perturbations)?

Lyapunov Functions?

Network´s Basin Stability

basin volume of a state (regime)

measures likelihood of return to this state (regime)

Nature Physics 9, 89 (2013)

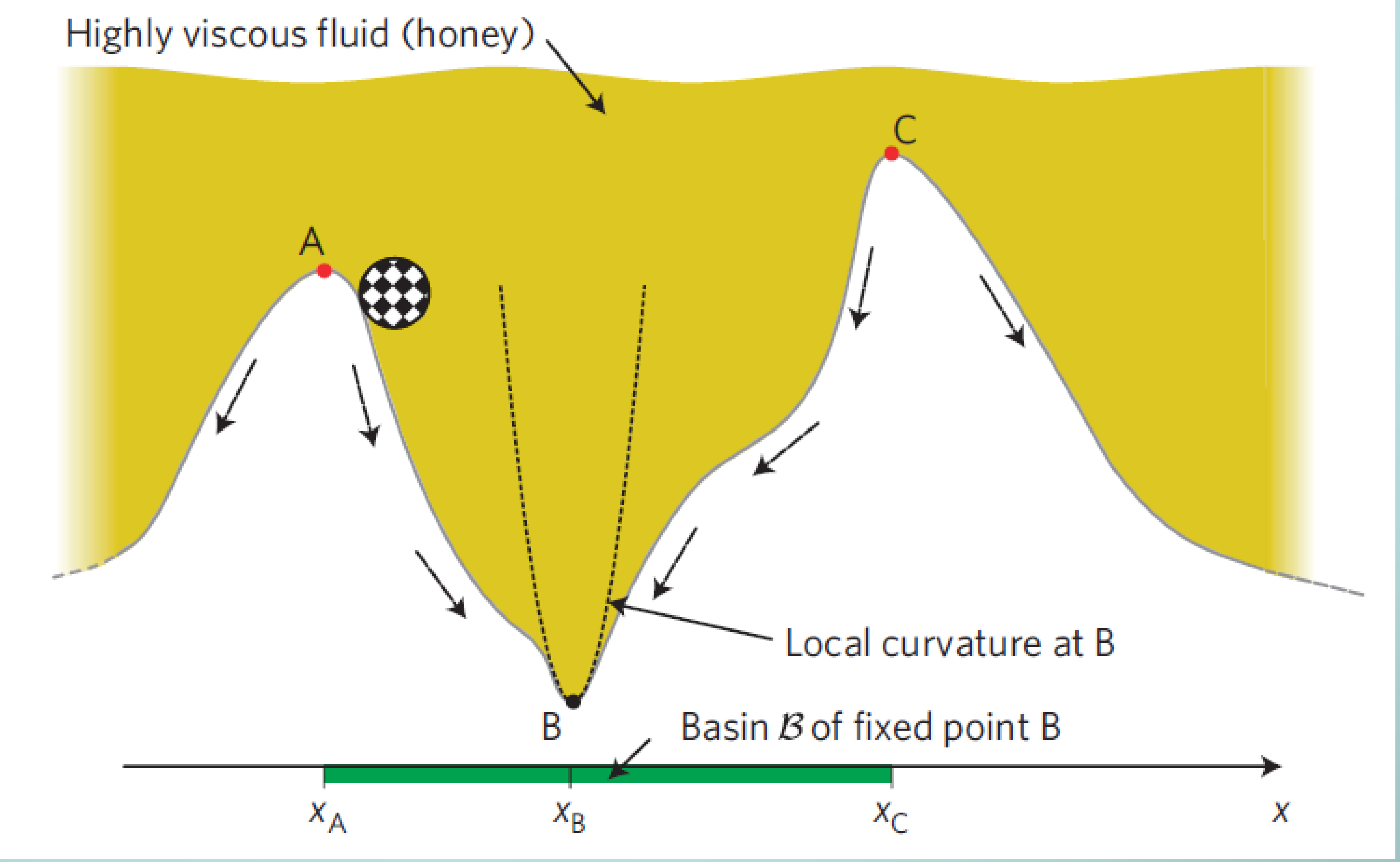


Figure 1 Thought experiment: marble on a marble track. The track is immersed in a highly viscous fluid to make the system's state space one-dimensional. Dashed arrows indicate where the marble would roll from each position. A, B and C label fixed points. Only B is stable. The green bar indicates B's basin of attraction B. If the marble is perturbed from B to a state within the basin, it will return to B. Such perturbations are permissible. Perturbations to states outside the basin are impermissible. The dashed parabola shows the local curvature around B, fitting the true marble track poorly in most of the basin.

Network´s Basin Stability

basin volume of a state (regime) measures the likelihood of

- arrival at this state (regime)
  quantifies its **relevance** (M. Girvan, 2006)
- return to this state after a random perturbation
  quantifies its **stability**
  (Menck, Heitzig, Marwan, Kurths:
  Nat. Phys., 2013)

Normalized Network´s Basin Stability

$\mathcal{B}$ - Synchronous state´s basin of attraction

$$\mathcal{B} = \{\mathrm{x} \in \mathcal{S} \,|\, \Phi_t(\mathrm{x}) \to \mathcal{I}\}$$

$\mathcal{Q}$ - Subset of state space S covering the system´s (weak) attractor

$$S_{\mathcal{B}\cap\mathcal{Q}} = \mathrm{Vol}(\mathcal{B} \cap \mathcal{Q})/\mathrm{Vol}(\mathcal{Q}) \in [0, 1]$$

Normalized Basin Stability

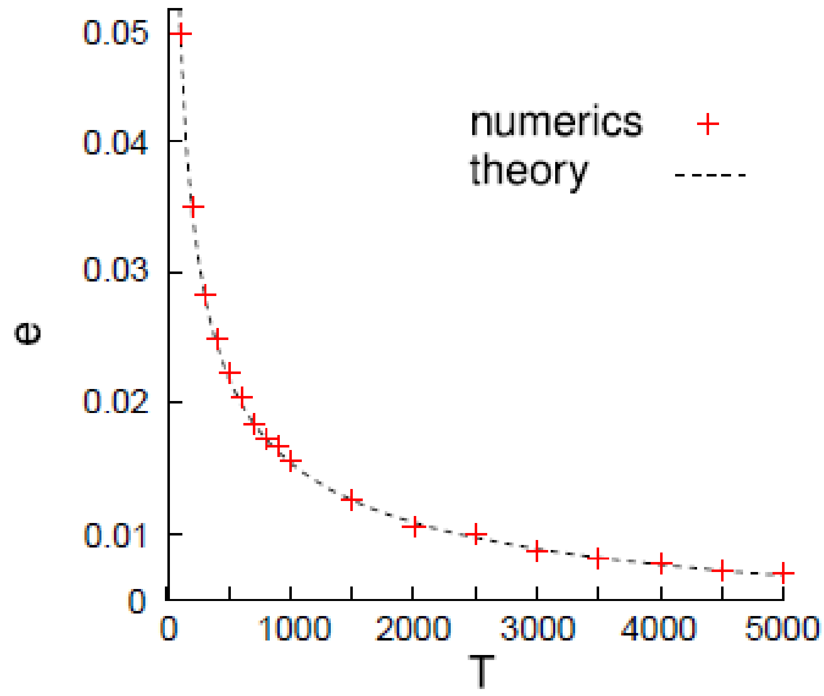
Bernoulli-like experiment

- T experiments (different initial conditions – randomly distributed)
- M states converge to $\mathcal{I}$
- Estimate M / T
  ➔ standard error $e := \frac{\sqrt{S_{\mathcal{B}}(1 - S_{\mathcal{B}})}}{\sqrt{T}}$

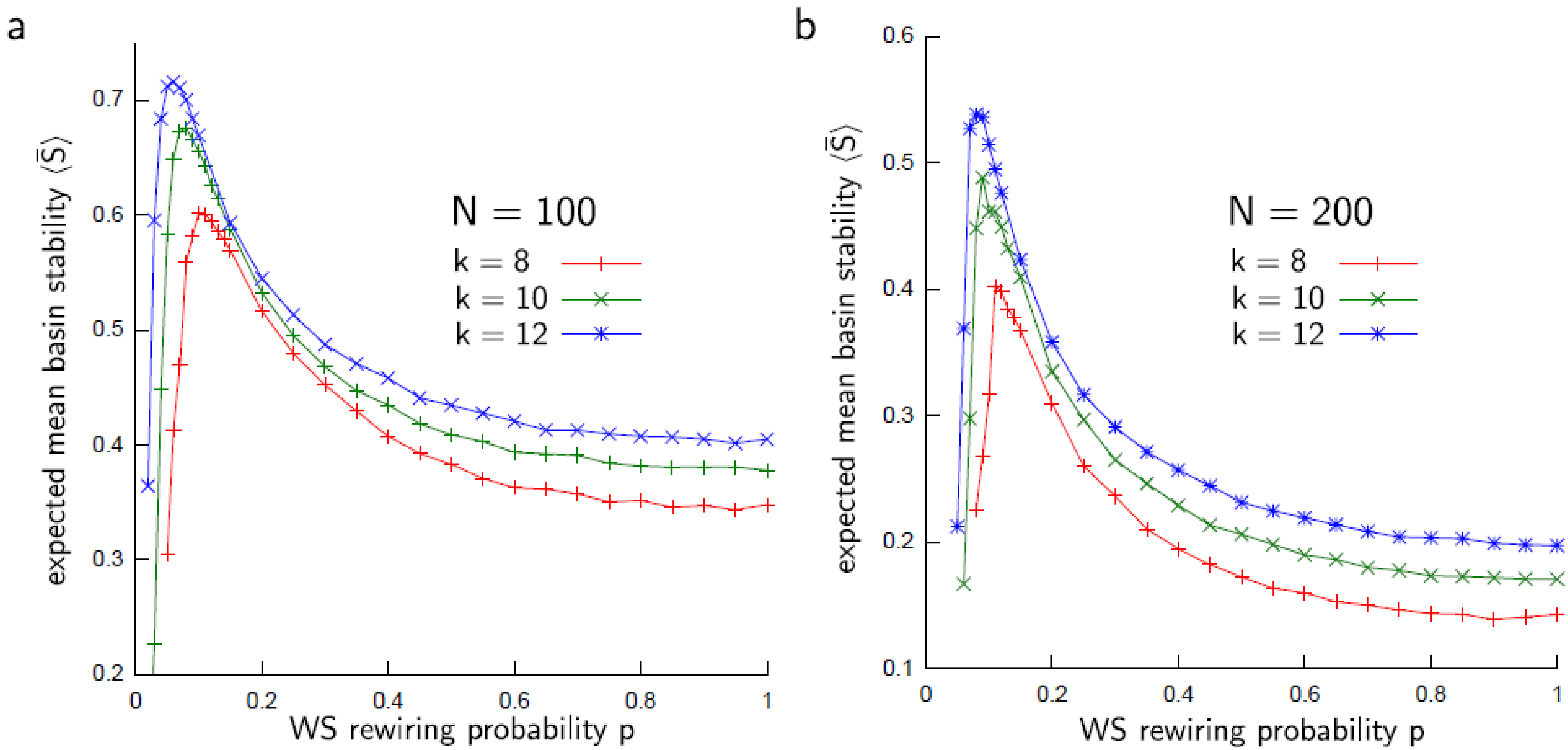
- T=500 ➔ error < 0.023

Basin Stability for the Rössler System

Q := q**N with $q = [-15, 15] \times [-15, 15] \times [-5, 35]$



: **Error of the basin stability estimation.** Red crosses indicate the standard error of the numerical basin stability estimation for different values of T. The dashed line shows the theoretical curve $e(T)$ as given by Eq. (2.35).

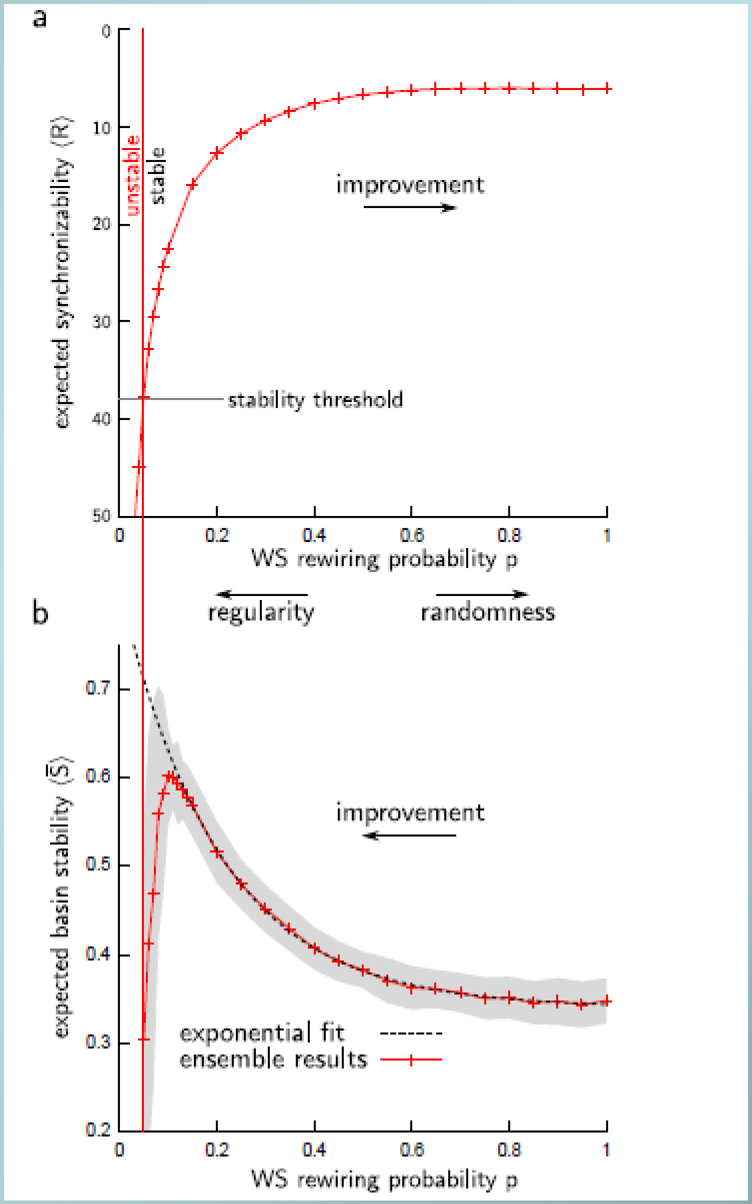


Supplementary Figure S1: **Basin Stability in Rössler networks.** Expected basin stability $\langle \bar{S} \rangle$ versus p. The grey shade indicates $\pm$ one standard deviation. The dashed line shows an exponential fitted to the ensemble results for $p \geq 0.15$. Solid lines are guides to the eye. **a**: $N = 100$, **b**: N=200.

$$\bar{S}_{\mathcal{B}} = \text{mean}_{K \in I_s} S_{\mathcal{B}}(K)$$

averaged over coupling strengths K

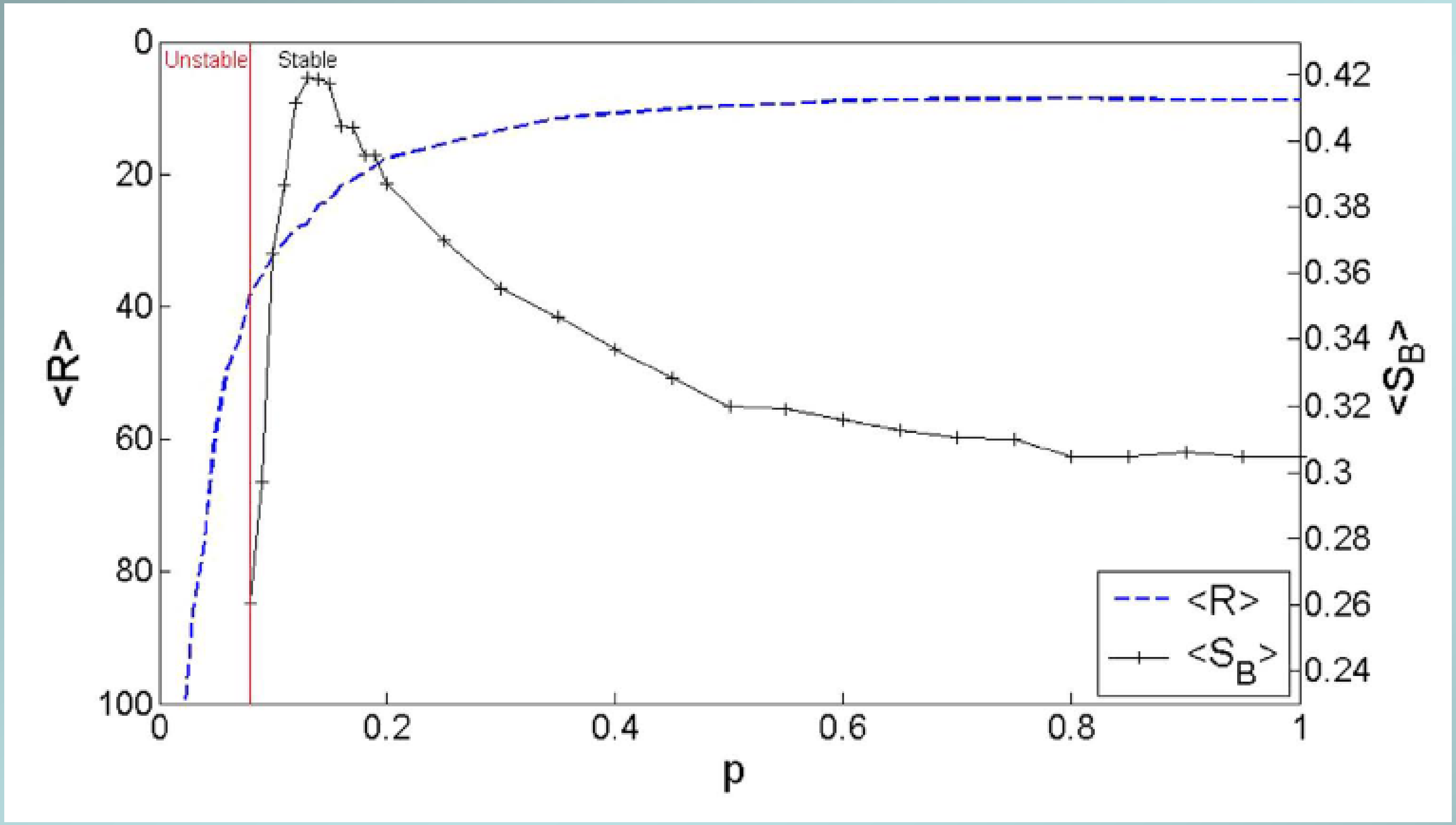
Synchronizability and basin stability inWatts-Strogatz (WS) networks
of chaotic oscillators.
a: Expected synchronizability R versus the WS model's parameter p.
The scale of the y-axis was reversed to indicate improvement upon increase in p.
b: Expected basin stability S versus p. The grey shade indicates one standard deviation.
The dashed line shows an exponential fitted to the ensemble results for p > 0.15. Solid lines are guides to the eye. The plots shown were obtained for N = 100 oscillators of Roessler type, each having on average k = 8 neighbours. Choices of larger N and different k produce results that are qualitatively the same.



Extension to delay-coupled systems

$$\dot{X}_i(t) = F[X_i(t)] - \sigma \sum_{j=1}^{N} g_{ij} h\left[X_j(t-\tau)\right],$$

Scient. Rep., 2016



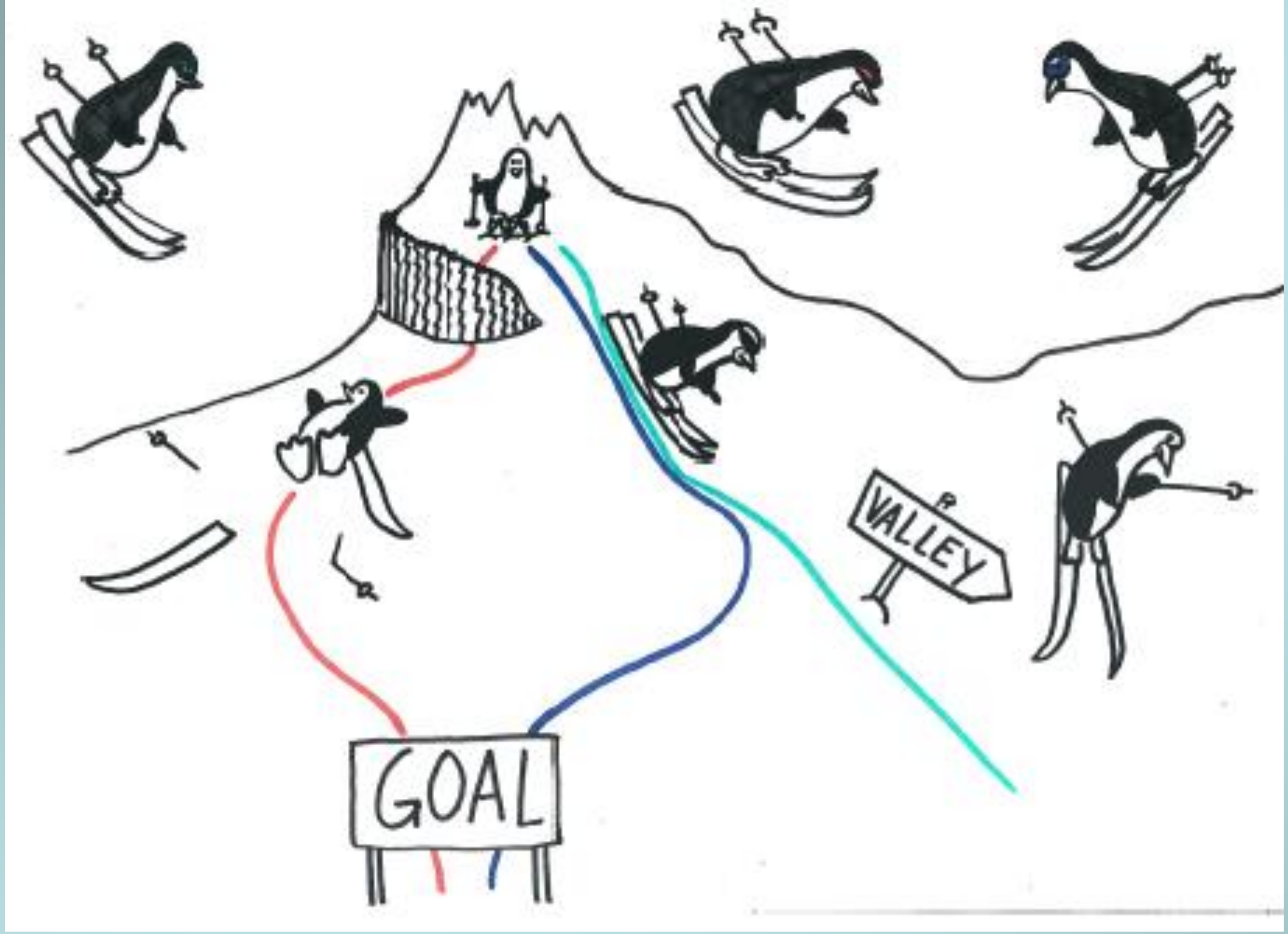
SW network, N = 100, chaotic Roessler oscillators, 6 neighbours each (in average) $\tau = 0.4.$

Other Approaches

- Basin stability refers to **asymptotic** behaviour and requires **multistability**
- In many applications (cybersystems, power grids, brain…) **transient** behaviour more important
- Apply concept of survivability

➔**Basin of Survival**

Scient. Rep. 2016



Penguin reaches goal in two way, but one via an **undesirable** state (due to cliff)

Basin of survival: all routes starting on top reaching the goal **savely**

Desirable region $X^+ \subset X$

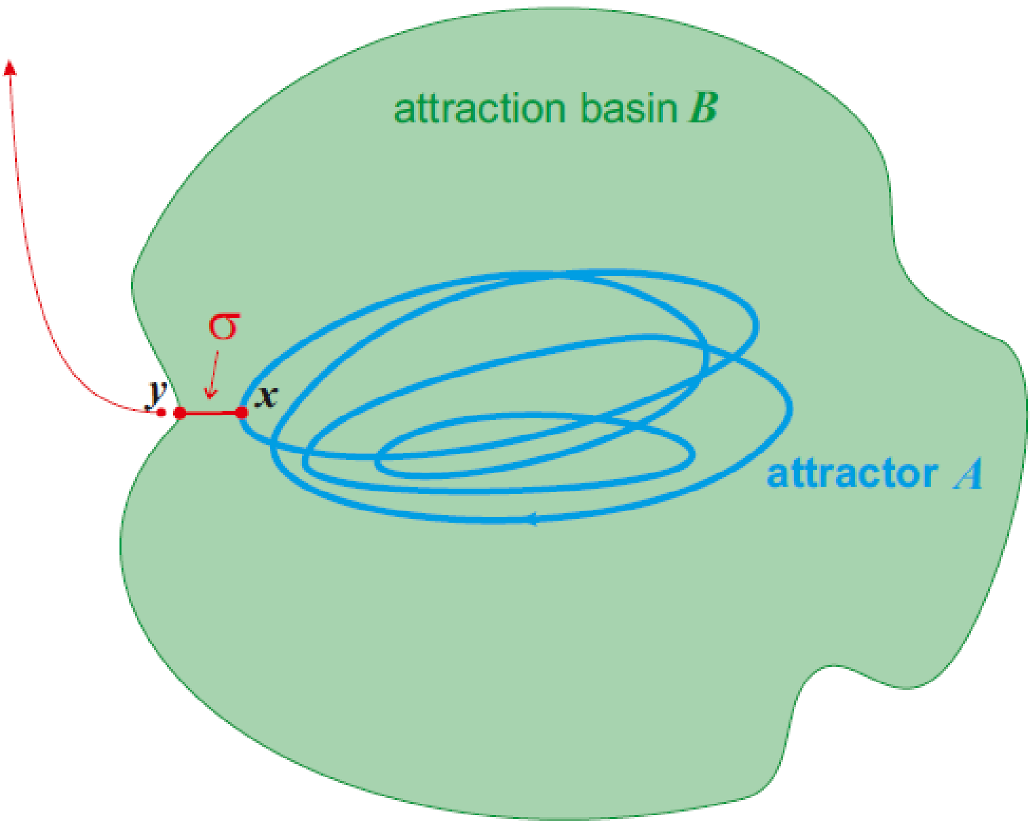
Survivability S(t):

Fraction of trajectories starting at X^+ and staying within X^+ the whole time [0 , t]

t-time basin of survival X_t^S

$$S(t) = \frac{\mathrm{Vol}(X_t^S)}{\mathrm{Vol}(X^+)}$$

Stability threshold

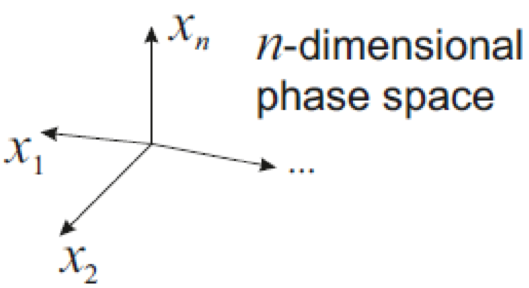


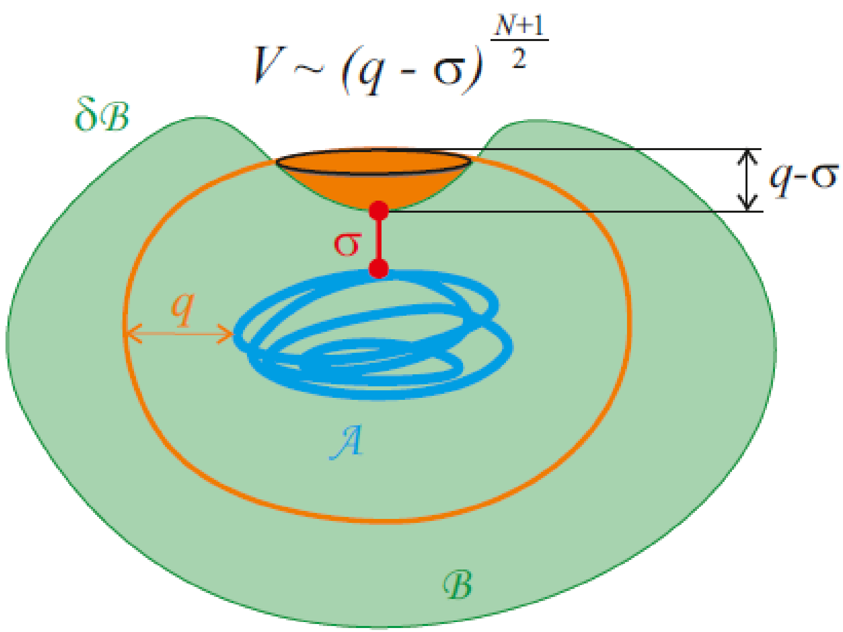
Stability threshold is the minimal perturbation kicking the system out of the attraction basin:

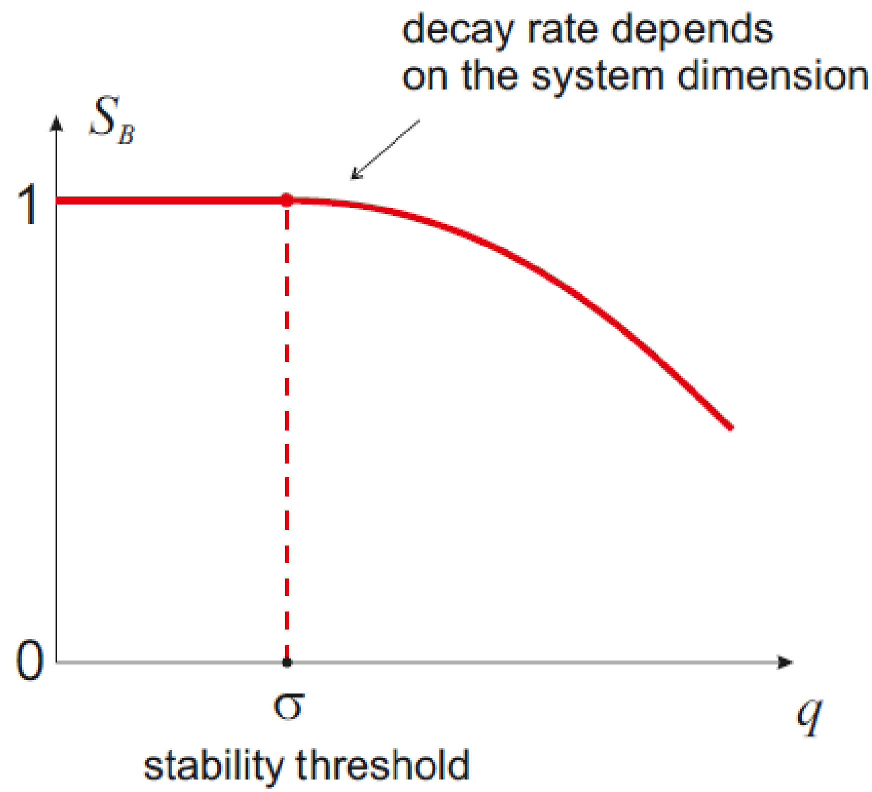
$$\sigma = \inf \{\mathrm{dist}(x,y) | x \epsilon \boldsymbol{A},\ y \epsilon \delta \boldsymbol{B}\}$$

New J. Physics, 2016

Stability threshold vs. basin stability

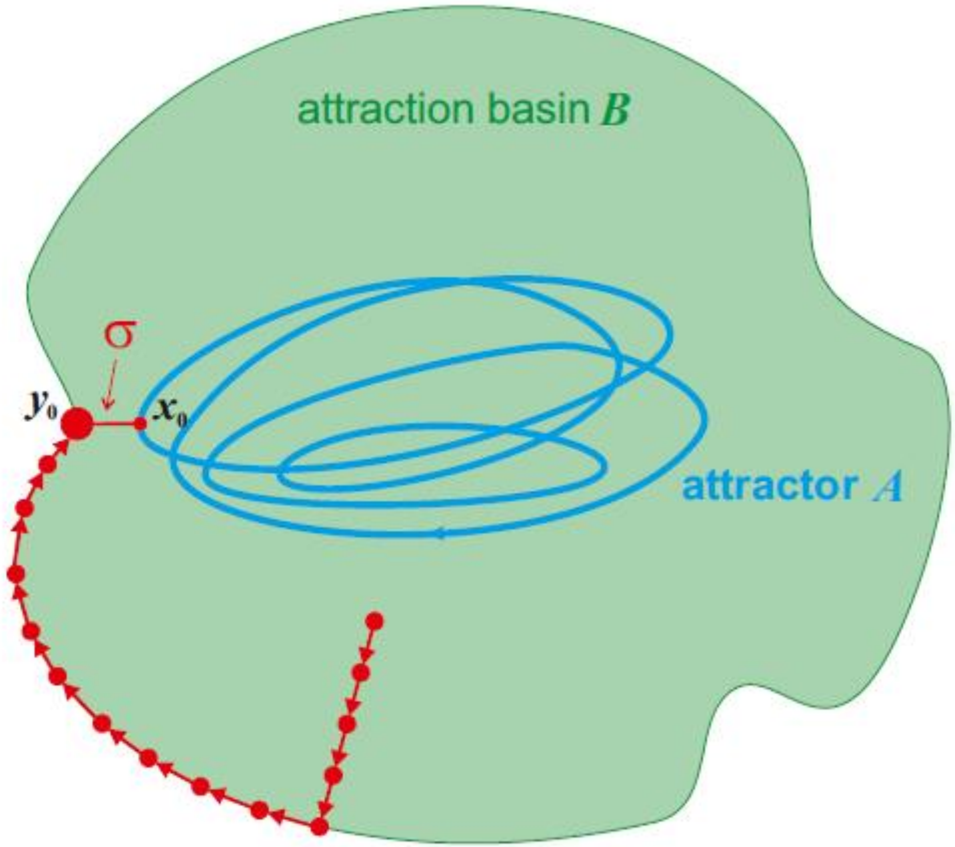






Perturbation class: magnitude < q

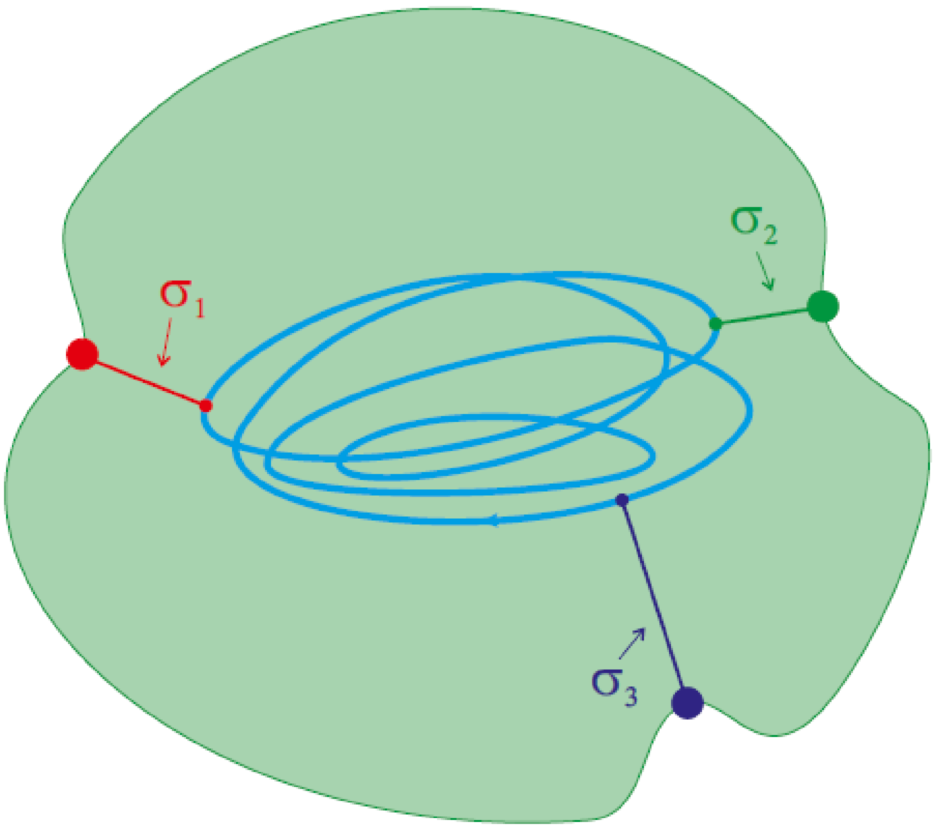
Stability threshold



Step 1. Reaching the border
Step 2. Moving along the border

$$\sigma = \mathrm{dist}(x_0, y_0)$$

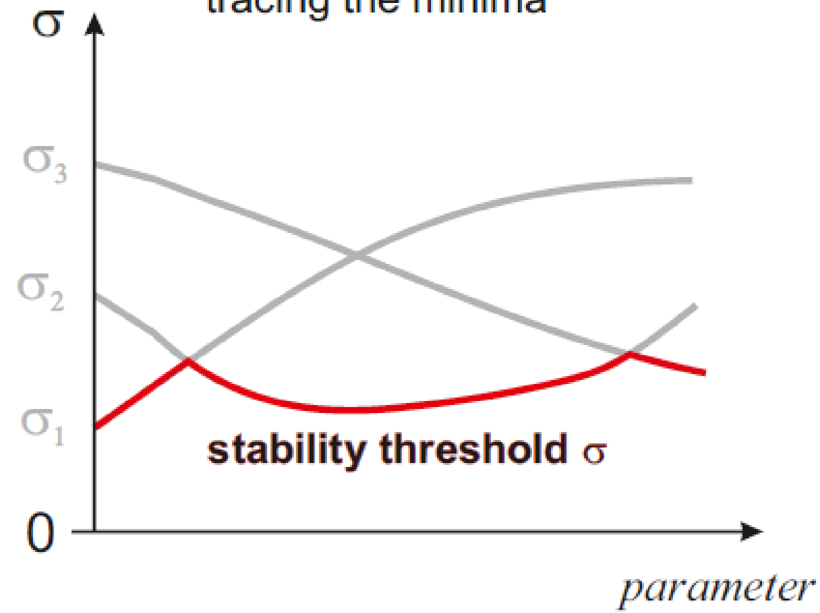
Global minimum vs. local minimums



Stability threshold corresponds to the global minimum:

$$\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$$

Variation of parameters: tracing the minima



Stochastic Basin of attraction (SBA) for metastable states

- SBA – set of initial conditions where solutions have a **small probability of exits** from the neighbourhood of an attractor and **high probability of returns** to it
- Calculation of these probabilities via elliptic partial differential eq. with Dirichlet boundary conditions

CHAOS 26, 073117 (2016)

Basic examples

- Three-well potential
- Genetic toggle system (Scient. Rep. 2016)
- Discontinuous systems (application: Amazonian vegetation) (Scient. Rep. 2017)
- Transport of particles in rough ratchets
- CO oxidation on Ir (111) surfaces

Finding special behaviour

- Le´vy noise with larger jumps and lower jump frequencies ($\alpha \leq 0.5$) enhances metastability (symmetric and asymmetric)
- Thermal noise stabilizes metastability in asymmetric potentials but reduces it in symmetric ones

Multistable mechanical systems

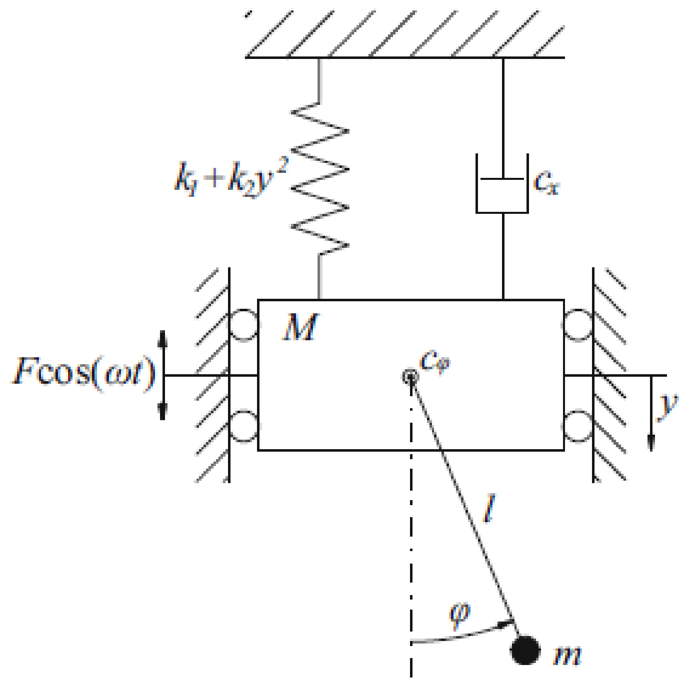


Fig. 1 The model of the first considered system. Externally forced Duffing oscillator with attached pendulum (tuned mass absorber)

$$\ddot{x} - ab\ddot{\gamma}\sin\gamma - ab\dot{\gamma}^2\cos\gamma + x + \alpha x^3 + d_1\dot{x} = f\cos\mu\tau,$$

$$\ddot{\gamma} - \frac{1}{b}\ddot{x}\sin\gamma + \sin\gamma + d_2\dot{\gamma} = 0,$$

Basin stability analysis

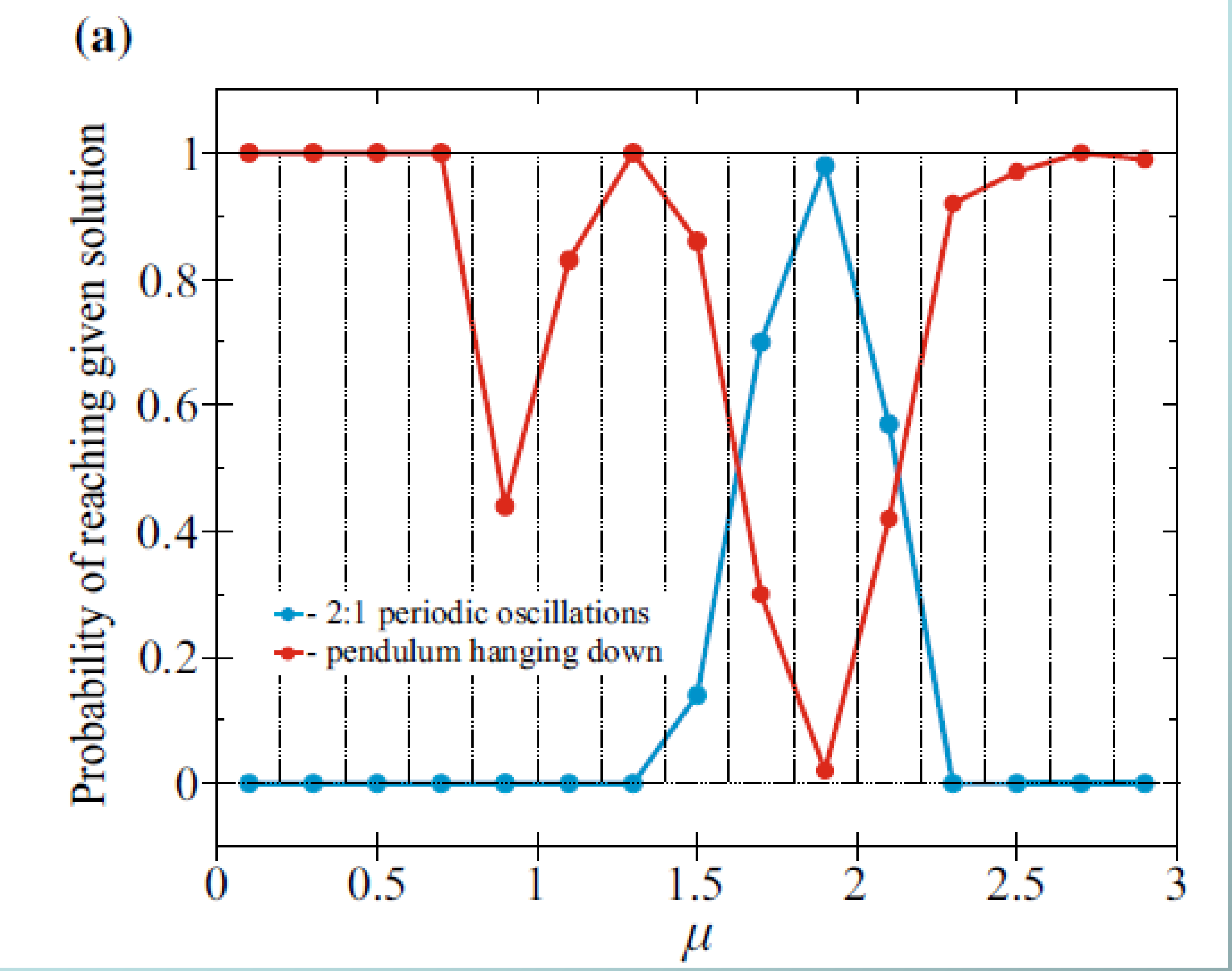
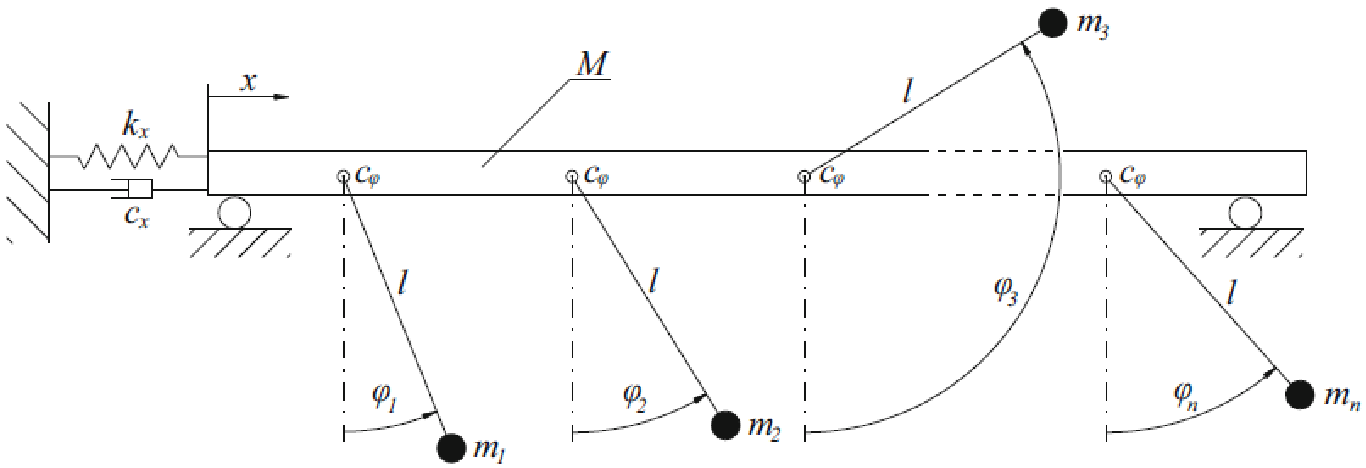


Fig. 3 The model of the third considered system. Horizontally moving beam with attached pendulums



$$m_i l^2 \ddot{\varphi}_i + m_i \ddot{x} l \cos\varphi_i + c_\varphi \dot{\varphi}_i + m_i g l \sin\varphi_i = N_0 - \dot{\varphi}_i N_1 \tag{2}$$

$$\left(M + \sum_{i=1}^{n} m_i\right)\ddot{x} + c_x \dot{x} + k_x x = \sum_{i=1}^{n} m_i l\left(-\ddot{\varphi}_i \cos\varphi_i + \dot{\varphi}_i^2 \sin\varphi_i\right) \tag{3}$$

BS Analysis

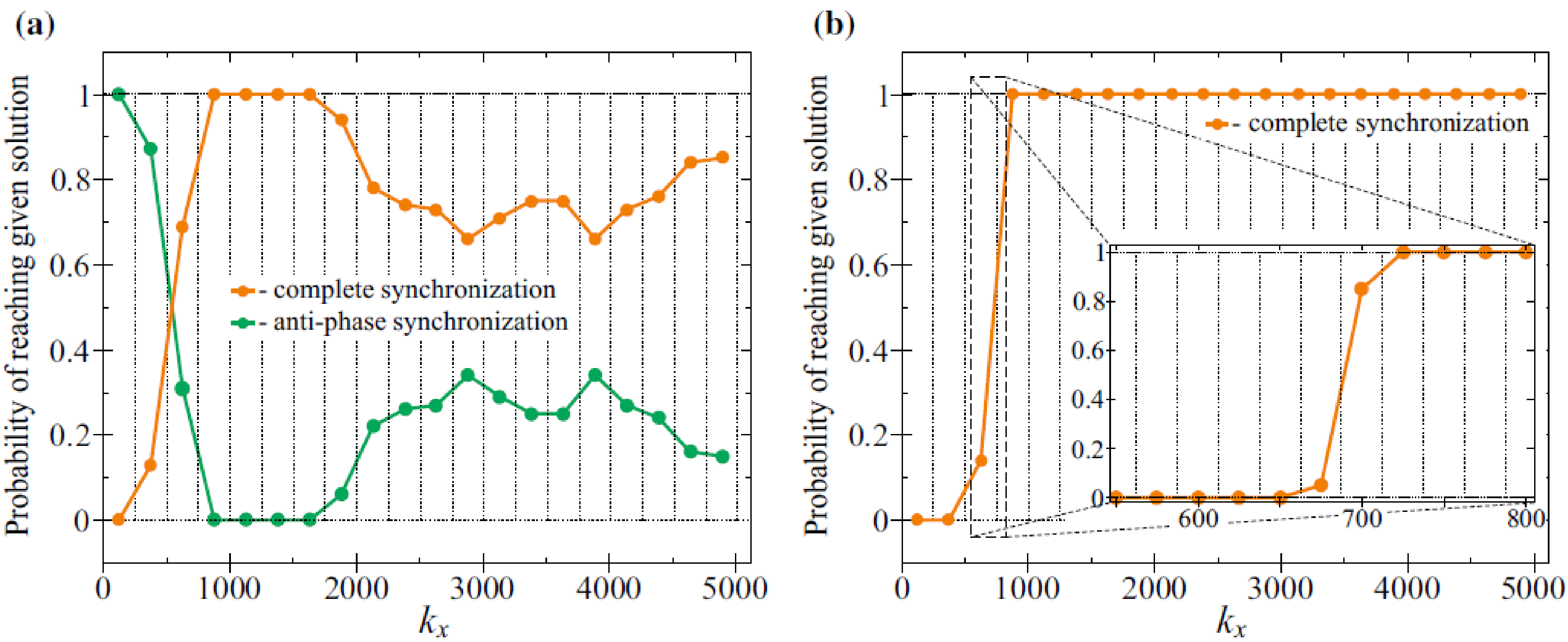


Fig. 10 Probability of reaching given solutions in the system with rotating pendula. Subplot **a** refers to the case with two pendula and **b** with twenty pendula. (Please note that on plot (**b**) and its magnification the initial conditions and parameter are somehow random, hence the results may slightly differ)

Multistable mechanical systems

- Generalized method for stability analysis

➔identification of parameter regions leading to a certain regime of basin stability

Meccanica 2016