Quasiperiodic Partial Synchrony in Oscillatory Networks

Michael Rosenblum and Arkady Pikovsky

Institute of Physics and Astronomy, Potsdam University, Germany

Pau Clusella, Antonio Politi

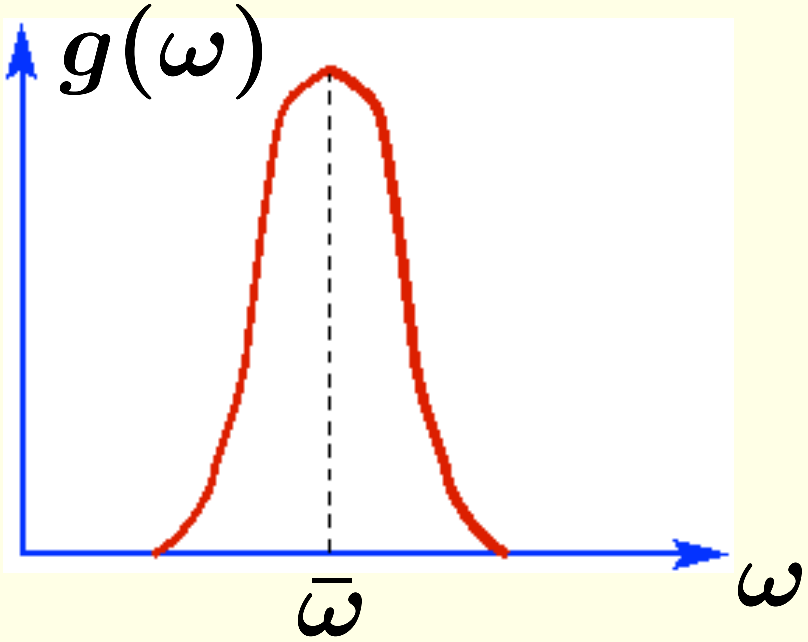
Amirkhan Temirbayev, Vladimir Ponomarenko

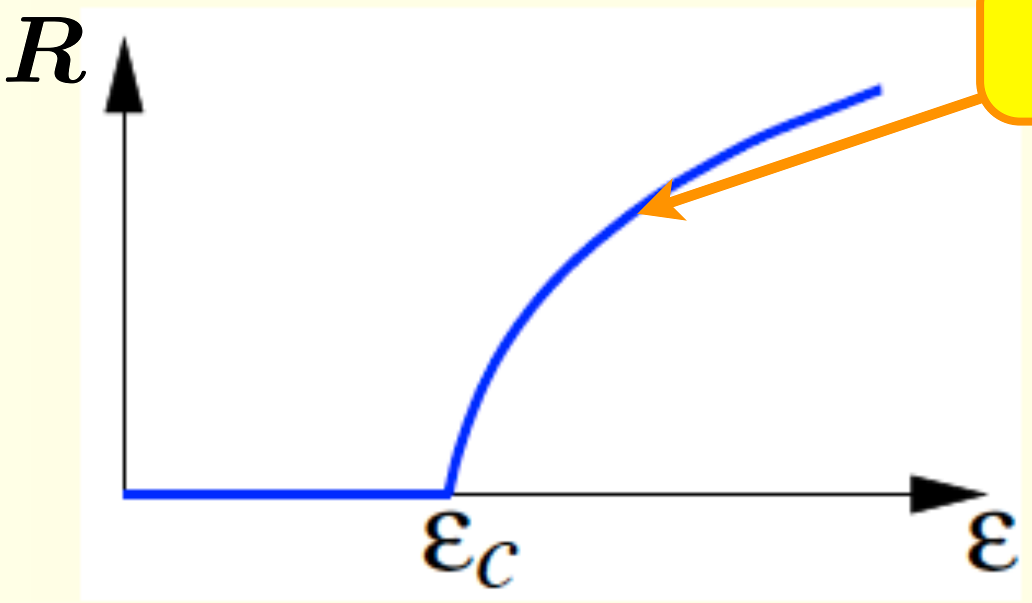
First International Summer Institute on Network Physiology, Como, Italy

Globally coupled oscillators: Synchronization transition

The Kuramoto model, oder parameter $R = N^{-1} \sum_j e^{i\varphi_j}$

$$\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k) \ , \quad k = 1, \ldots, N$$





$$R \sim \sqrt{\varepsilon - \varepsilon_c}$$

Critical coupling:

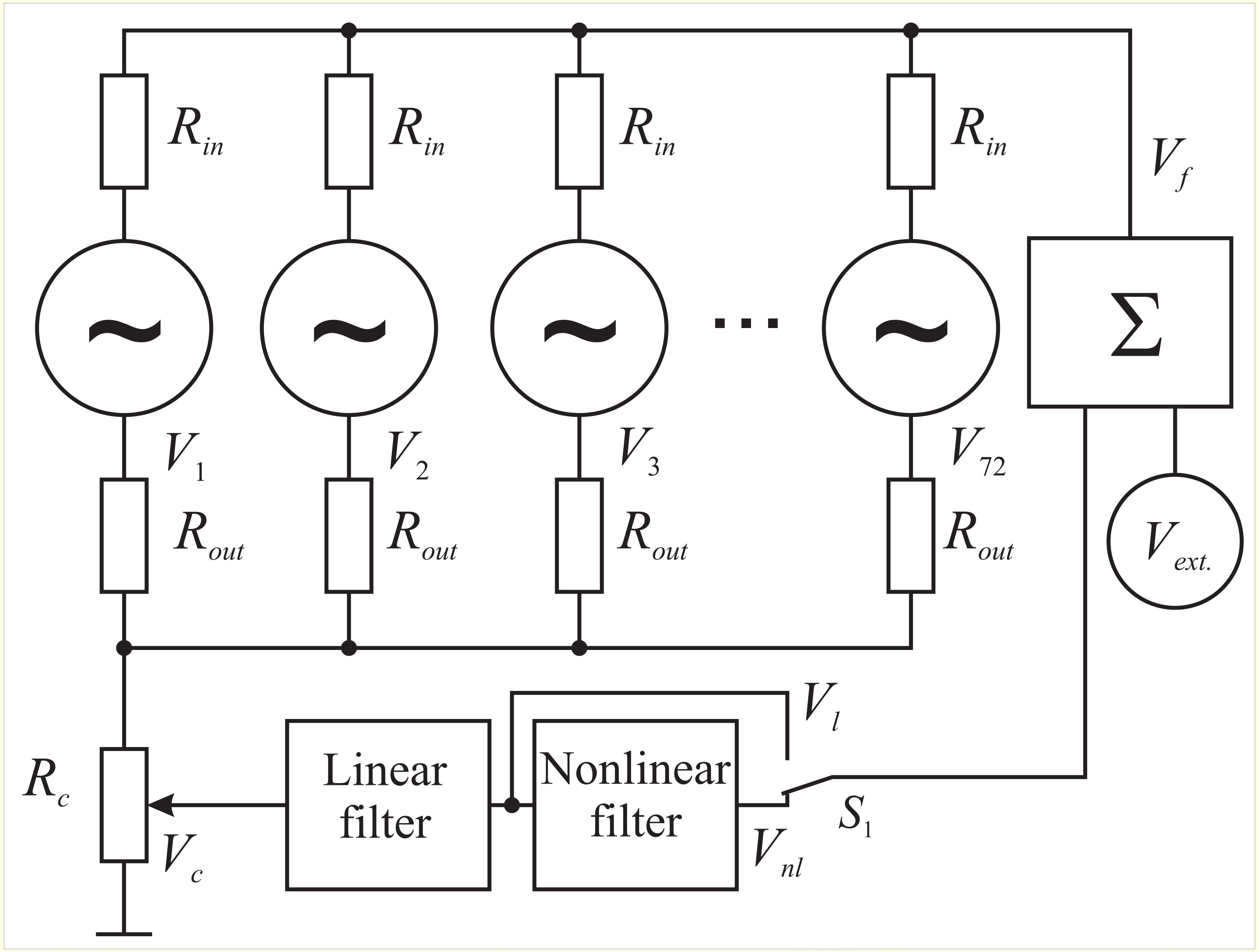
$$\varepsilon_c = \frac{2}{\pi g(\bar{\omega})}$$

- Unimodal frequency distribution: ~ 2nd order phase transition
- Uniform frequency distribution: ~ 1st order phase transition

(D. Pazó, 2005)

Experiments on electronic circuits

72 oscillators

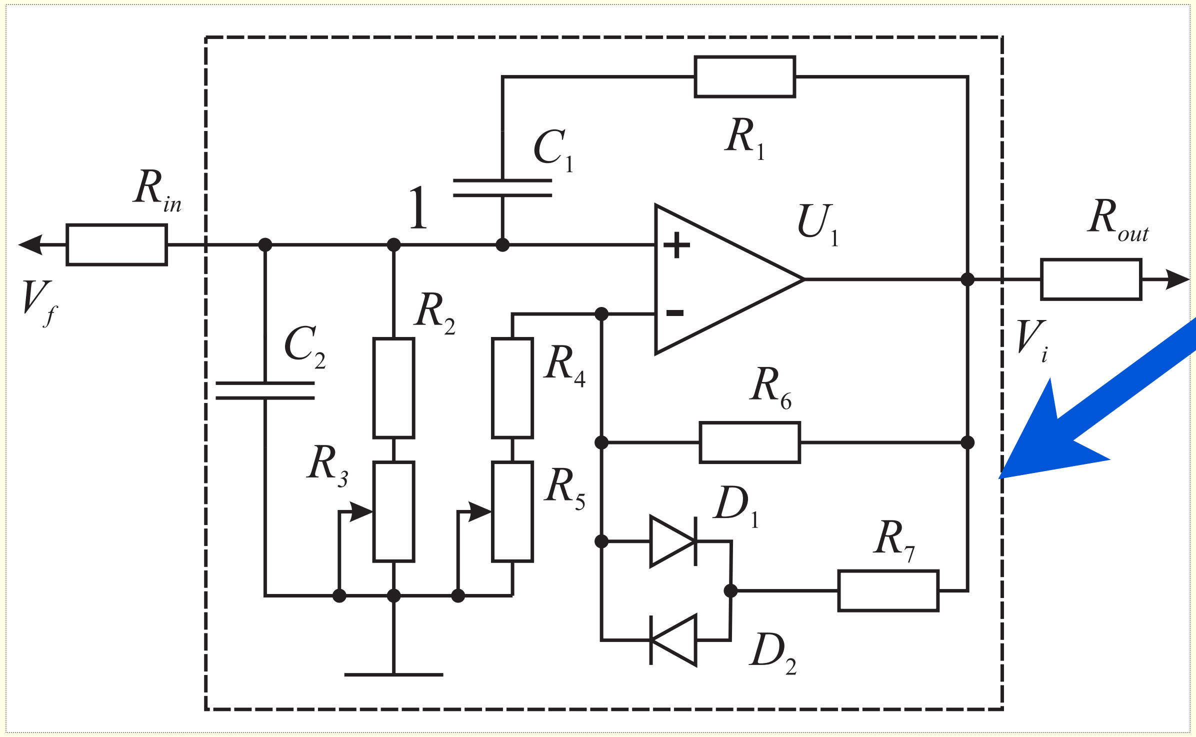


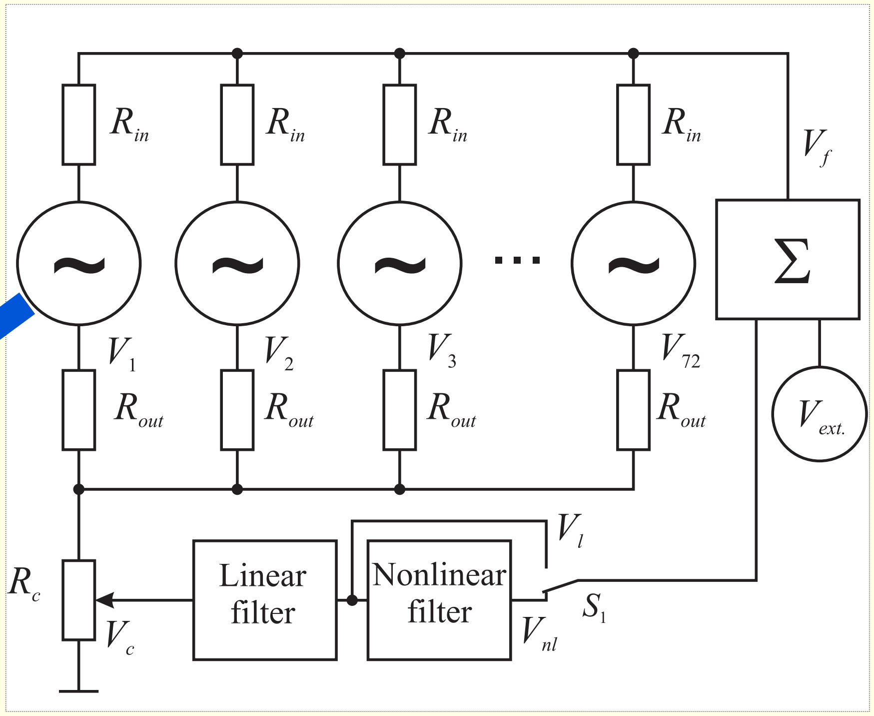
mean field coupling $V_c \approx \varepsilon \frac{\sum_{i=1}^{N} V_i}{N}$, with $0 \le \varepsilon \le 1$

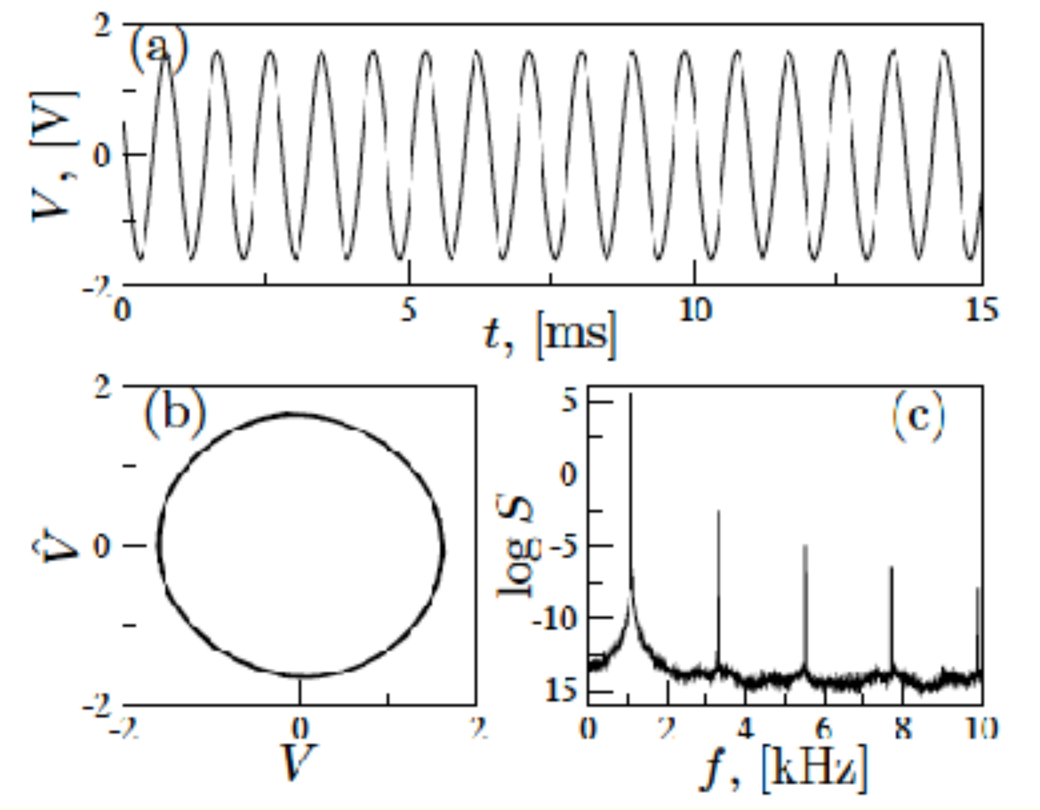
Temirbayev et al., Phys. Rev E 2012, 2013

Individual unit

Wien-bridge oscillator



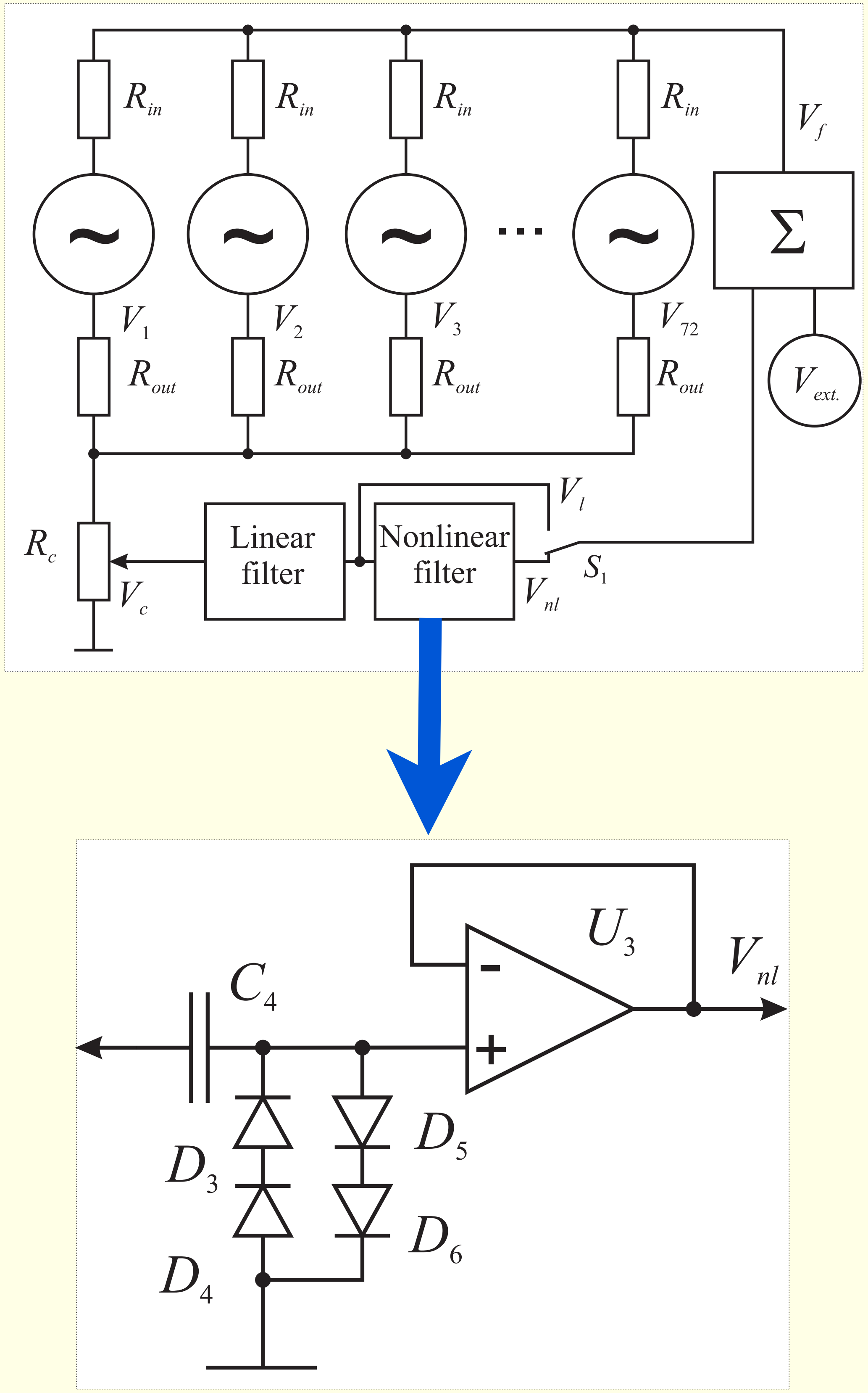




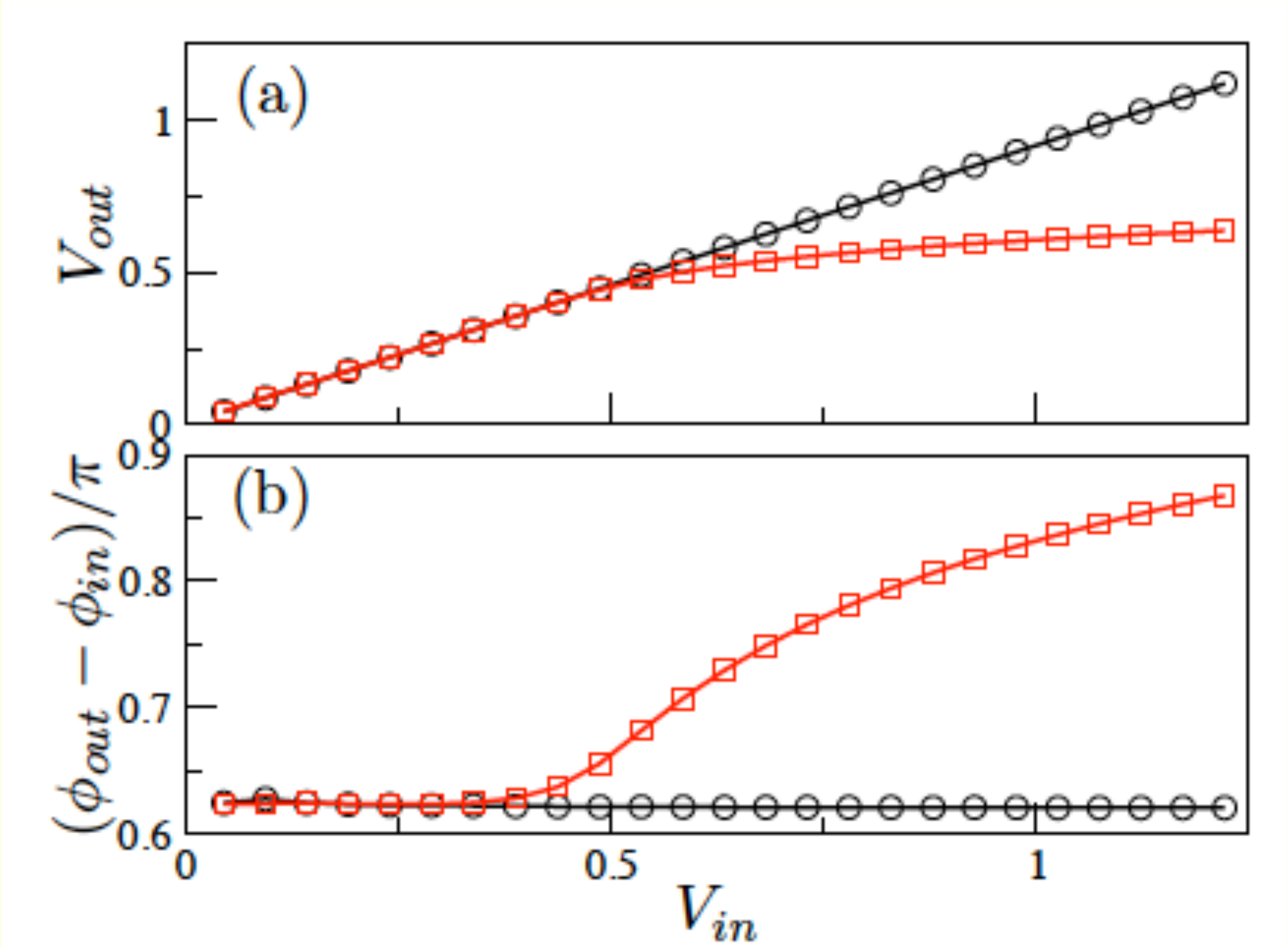
van der Pol-type equation

$$\ddot{u} - \mu(1 - \alpha u^2 + \beta u^4)\dot{u} + \Omega^2 u = \nu \dot{V}_f + \nu\omega V_f$$

Experiments on electronic circuits



Nonlinear dependence of the phase shift on the amplitude of the mean field

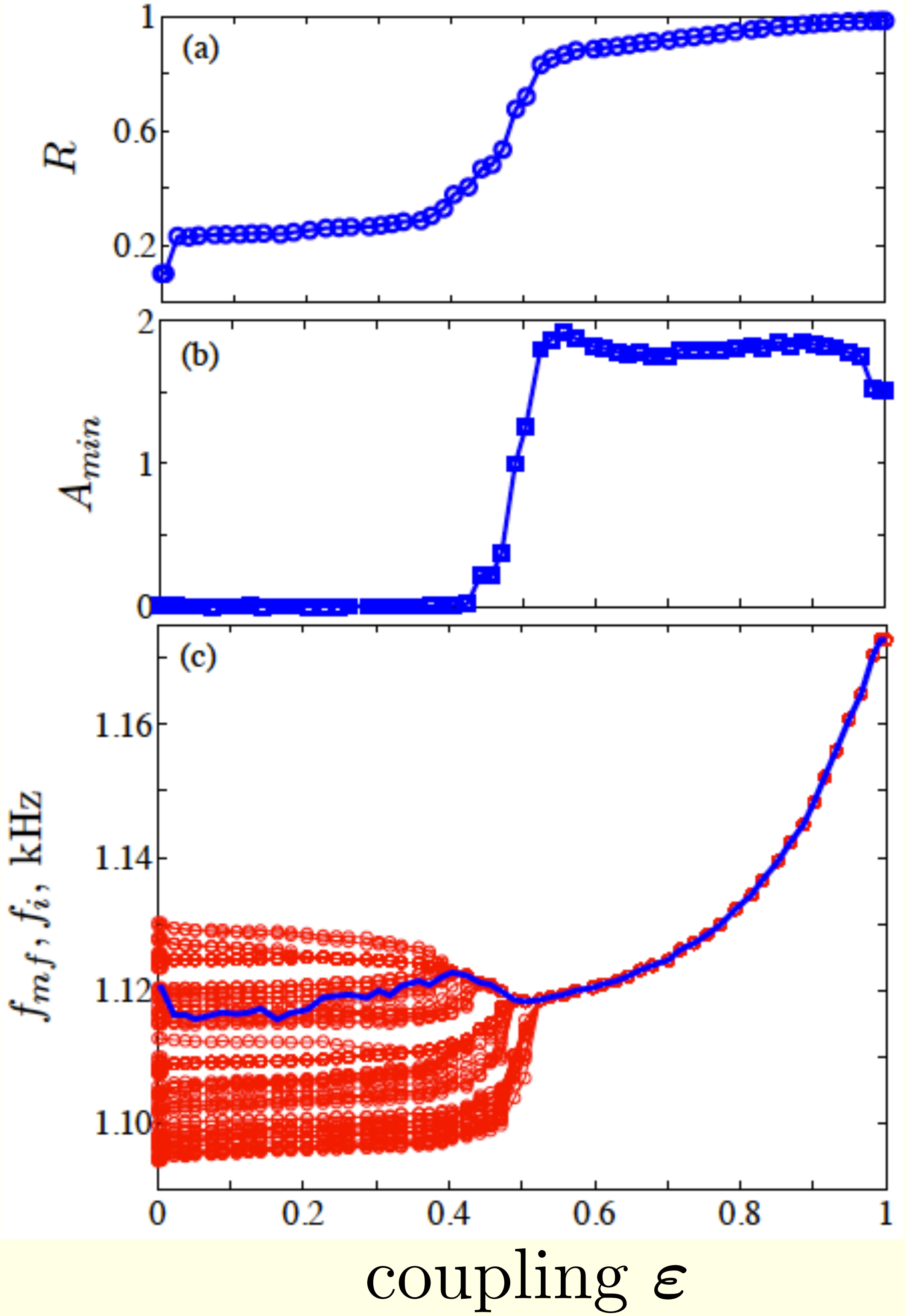


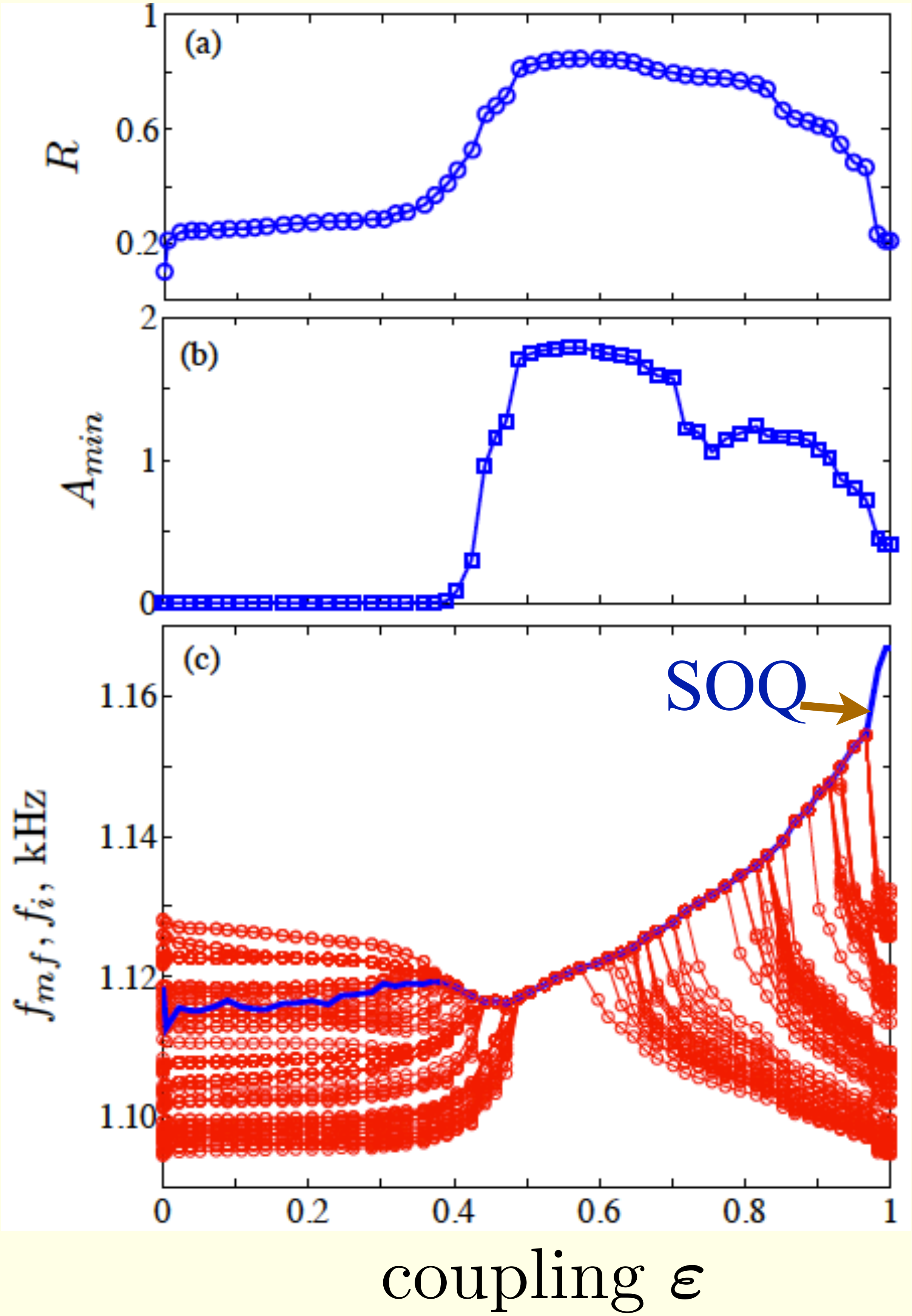
red: nonlinear filter, black: linear filter

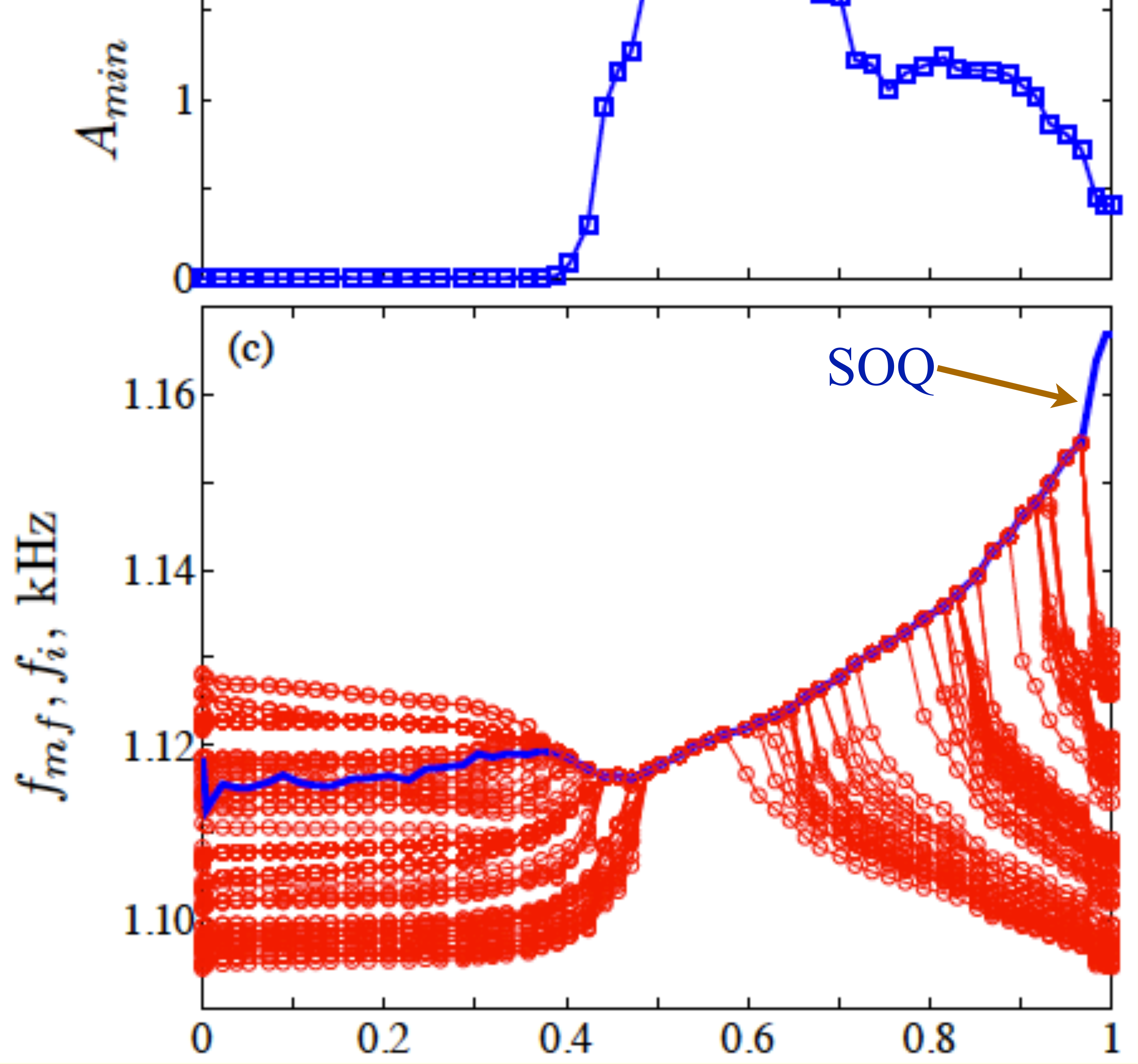
Measurements and data analysis

- 73 channels (outputs of all oscillators + mean field)
- Sampling frequency 20 kHz, 5*10^4 points
- 5 measurements for each value of the coupling strength ε
- Overall: approx. 12500 oscillation periods
- Phase and frequency determination via the Hilbert Transform
- Time-averaged order parameter $R = \left\langle N^{-1} \left| \sum_k \exp(i\phi_k) \right| \right\rangle_t$
- Minimal mean field amplitude $A_{min} = \min_t A(t)$

Results: linear vs nonlinear phase-shifting unit

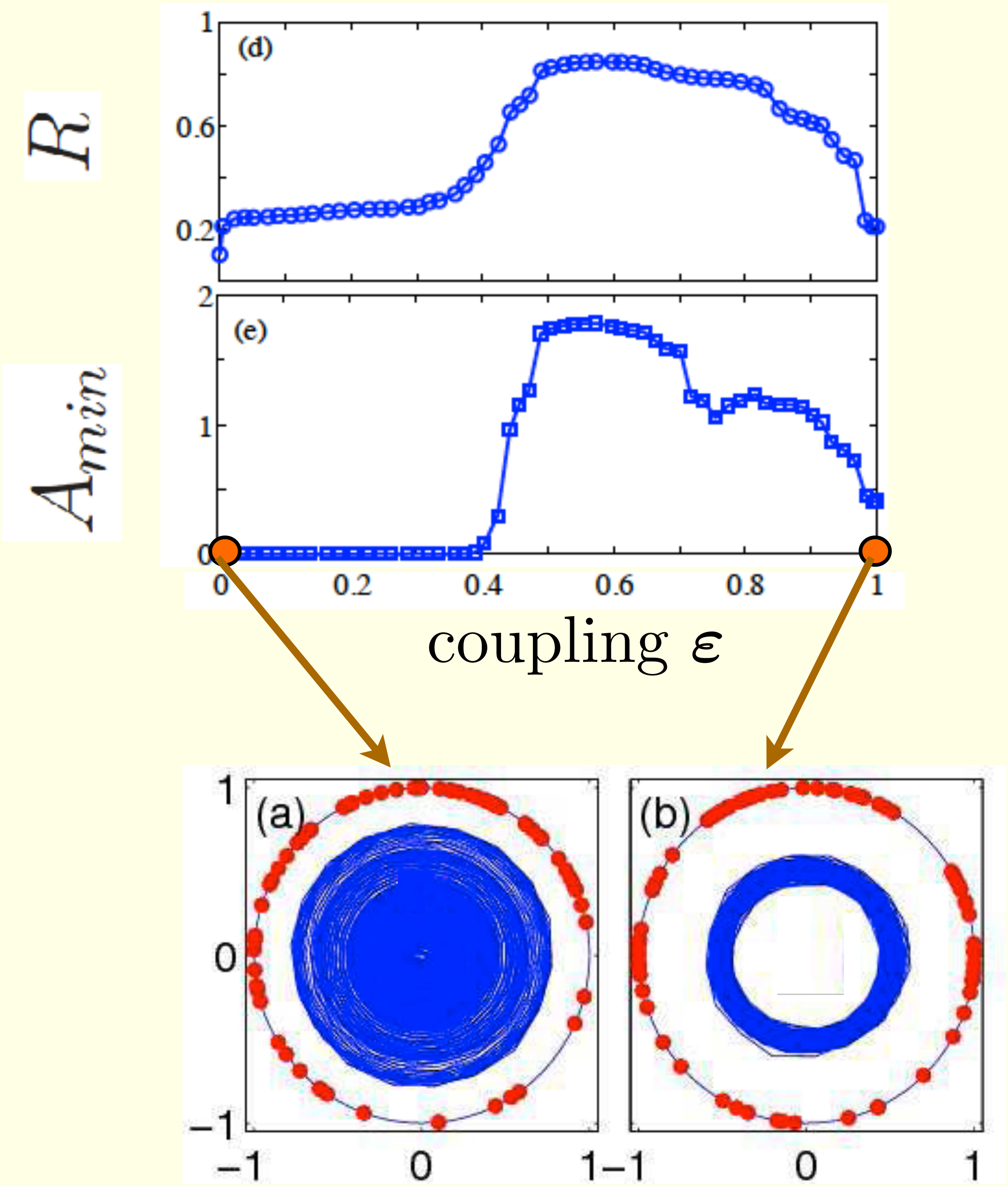




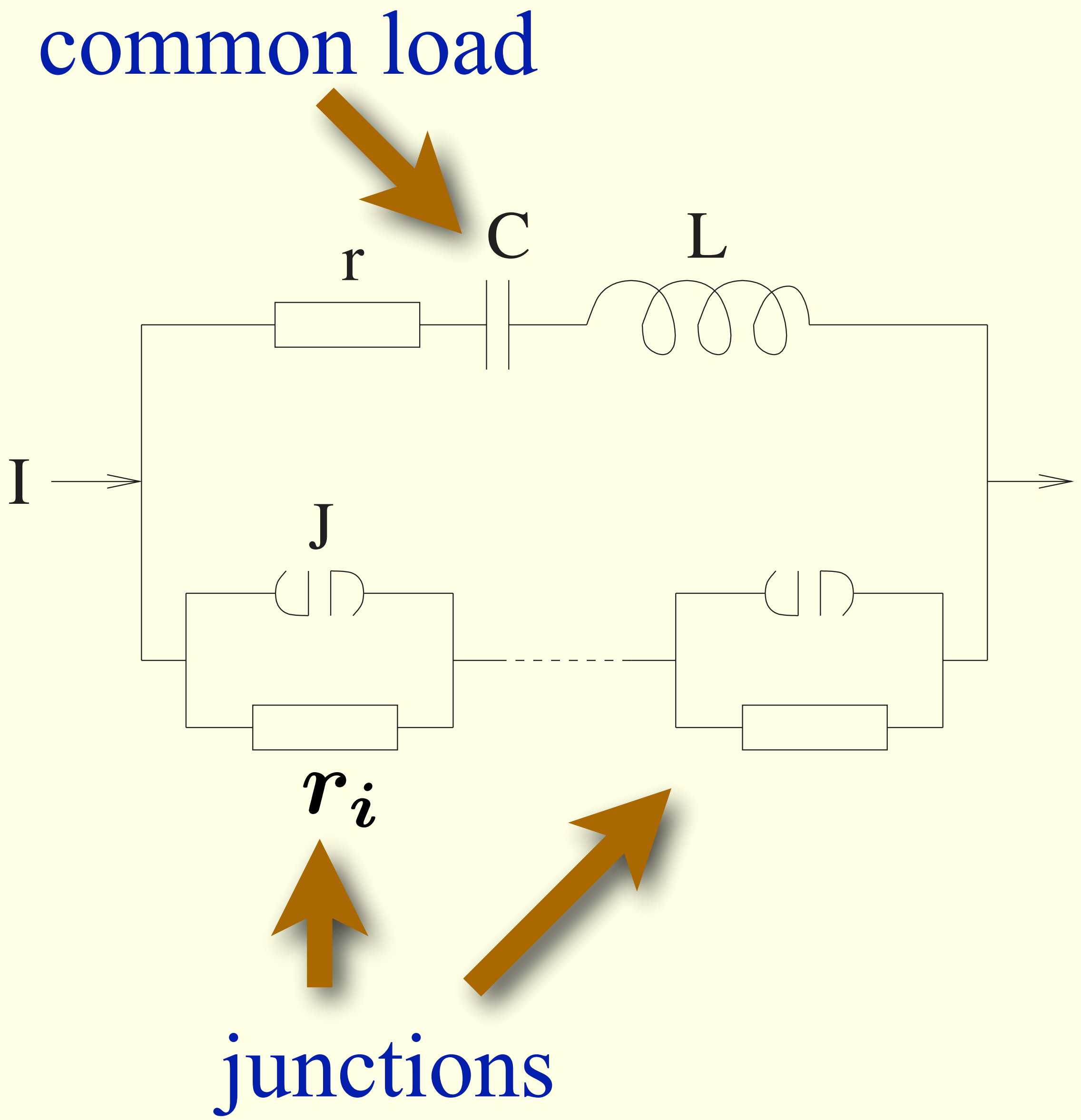


coupling ε

Minimal amplitude as indicator of coherence



Numerical example: Josephson junctions array



Wiesenfeld & Swift 1995

Junctions are coupled via a common load (*LrC*-circuit)

$$\frac{\hbar}{2er_i}\frac{d\Psi_k}{dt}+I_c\sin\Psi_k=I-\frac{dQ}{dt}$$

$$L\frac{d^2Q}{dt^2}+r\frac{dQ}{dt}+\frac{Q}{C}=\frac{\hbar}{2e}\sum_j\frac{d\Psi_j}{dt}$$

Here global coupling has its own dynamics

For weak coupling the model reduces to the Kuramoto model

Nonlinear common load

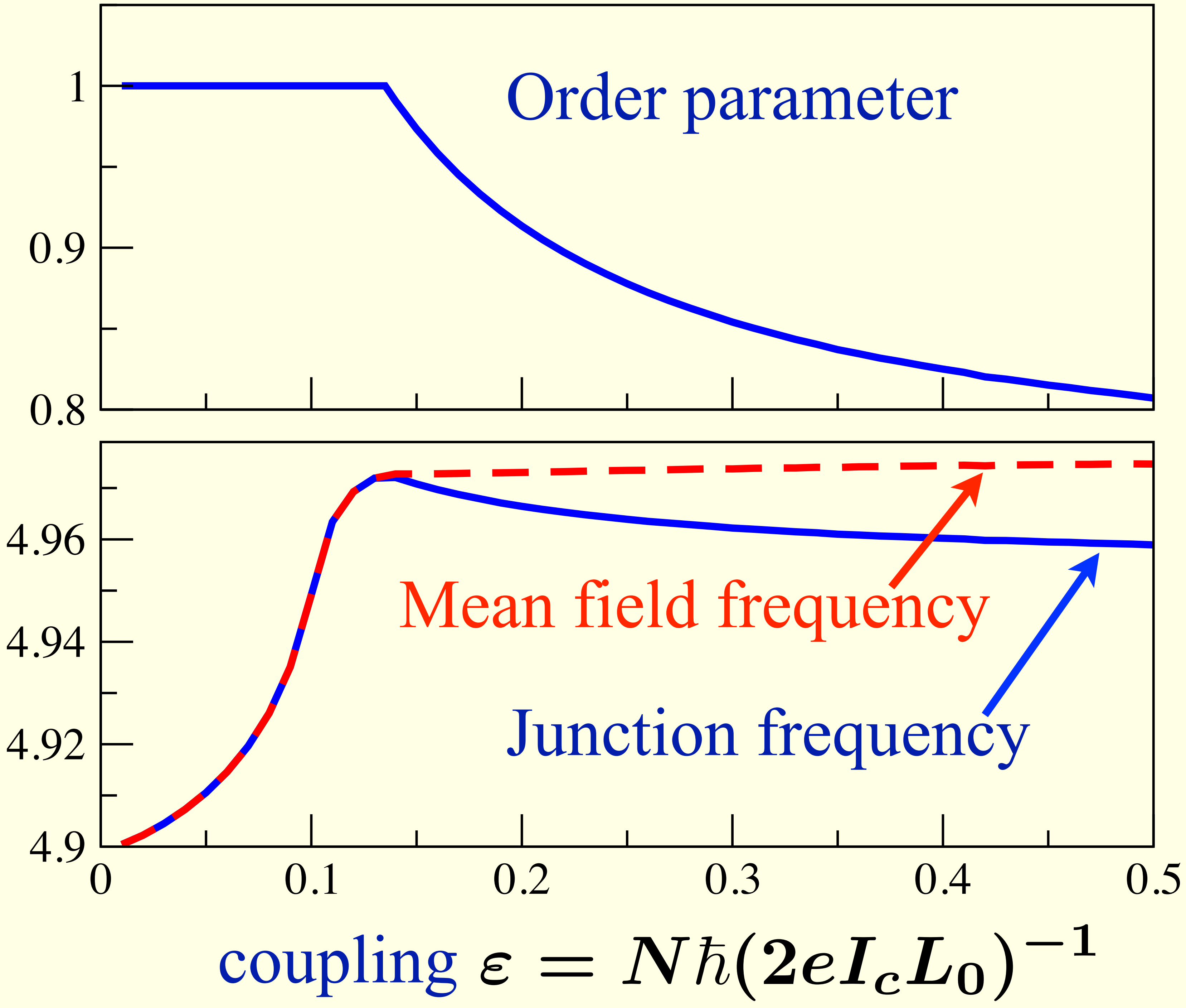
Josephson junctions, coupled via a *LrC*-circuit with nonlinear inductance (Rosenblum & Pikovsky, Phys. Rev. Lett. 2007)

Magnetic flux Φ nonlinearly depends on the current $\dot{Q}$

$$L\frac{d\Phi}{dt} + r\frac{dQ}{dt} + \frac{Q}{C} = \frac{\hbar}{2e}\sum_j \frac{d\Psi_j}{dt}\;, \qquad \Phi = L_0\dot{Q} + L_1\dot{Q}^3$$

identical units

Nonlinearly coupled Josephson junctions: numerics



- Breakup of synchrony for $\varepsilon > \varepsilon_{crit}$
- Mean field is faster than individual units

Frequencies are incommensurate

↓

Quasiperiodic dynamics

(Rosenblum & Pikovsky, PRL 2007)

Identical globally coupled oscillators

Consider the simplest network:

- Elements are identical and subject to common force
- Phase oscillators, one-harmonic coupling

→ The paradigmatic Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta),\ \mathit{with}\ R e^{i\Theta} = \frac{1}{N}\sum_j e^{i\varphi_j}$$

$|\beta| < \pi/2$ → attractive coupling

$\pi/2 < \beta < 3\pi/2$ → repulsive coupling

$\beta = \pm\pi/2$ → neutral coupling

The Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta)$$

$|\beta| < \pi/2$ → synchrony is stable

$\pi/2 < \beta < 3\pi/2$ → splay state is stable

$\beta = \pm\pi/2$ → marginal stability
(not interesting)

either full synchrony, $R = 1$
or full asynchrony (splay state), $R = 0$

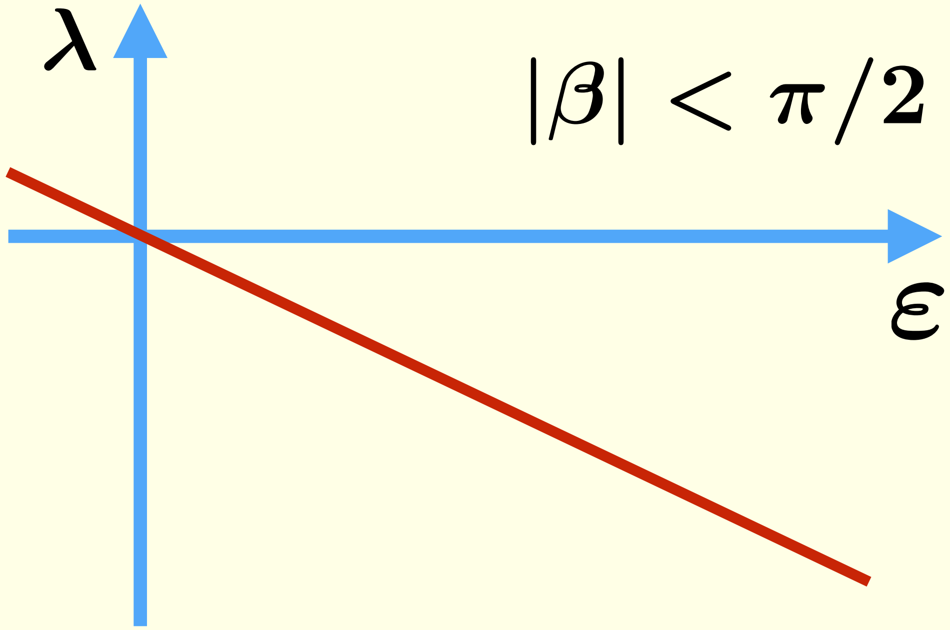
Notice: clusters are not possible, as follows from the Watanabe-Strogatz theory (except for ***N-1,1*** configuration)

Stability of the synchronous state

The Kuramoto-Sakaguchi model, identical oscillators:

Synchronous (one-cluster) state is stable, if $\boldsymbol{\lambda} = -\varepsilon \cos \beta < 0$

eigenvalue



For this model: stability is proportional to coupling

==> tendency to synchrony increases with ε

Specific features of the Kuramoto-Sakaguchi model

1) tendency to synchrony increases with the coupling strength
2) domains of stable synchrony and asynchrony are **complementary**
3) only full synchrony or splay state; no clusters, no chimeras

These properties are typical, but not general!

General phase models:
When do we expect complex solutions?

1) tendency to synchrony is not monotonic and/or

2) **both** splay state and synchrony are unstable

The system settles at some intermediate state

We expect: clusters

chimeras

quasiperiodic partially synchronous states

Quasiperiodic partial synchrony in the Kuramoto-Daido model

1) continuous but not uniform distribution of phases

→ order parameter $0 < R < 1$

2) Mean field frequency $\neq$ oscillators frequency

→ quasiperiodic dynamics

To be distinguished from the case of ensembles with a frequency distribution, when some oscillators form a synchronous cluster while some are not locked to the mean field

A minimal model

Kuramoto-Daido model with two harmonics, Hansel et al, 1993

$$\dot{\varphi}_k = R_1 \sin(\Theta_1 - \varphi_k + \gamma_1) + aR_2 \sin(\Theta_2 - 2\varphi_k + \gamma_2)$$

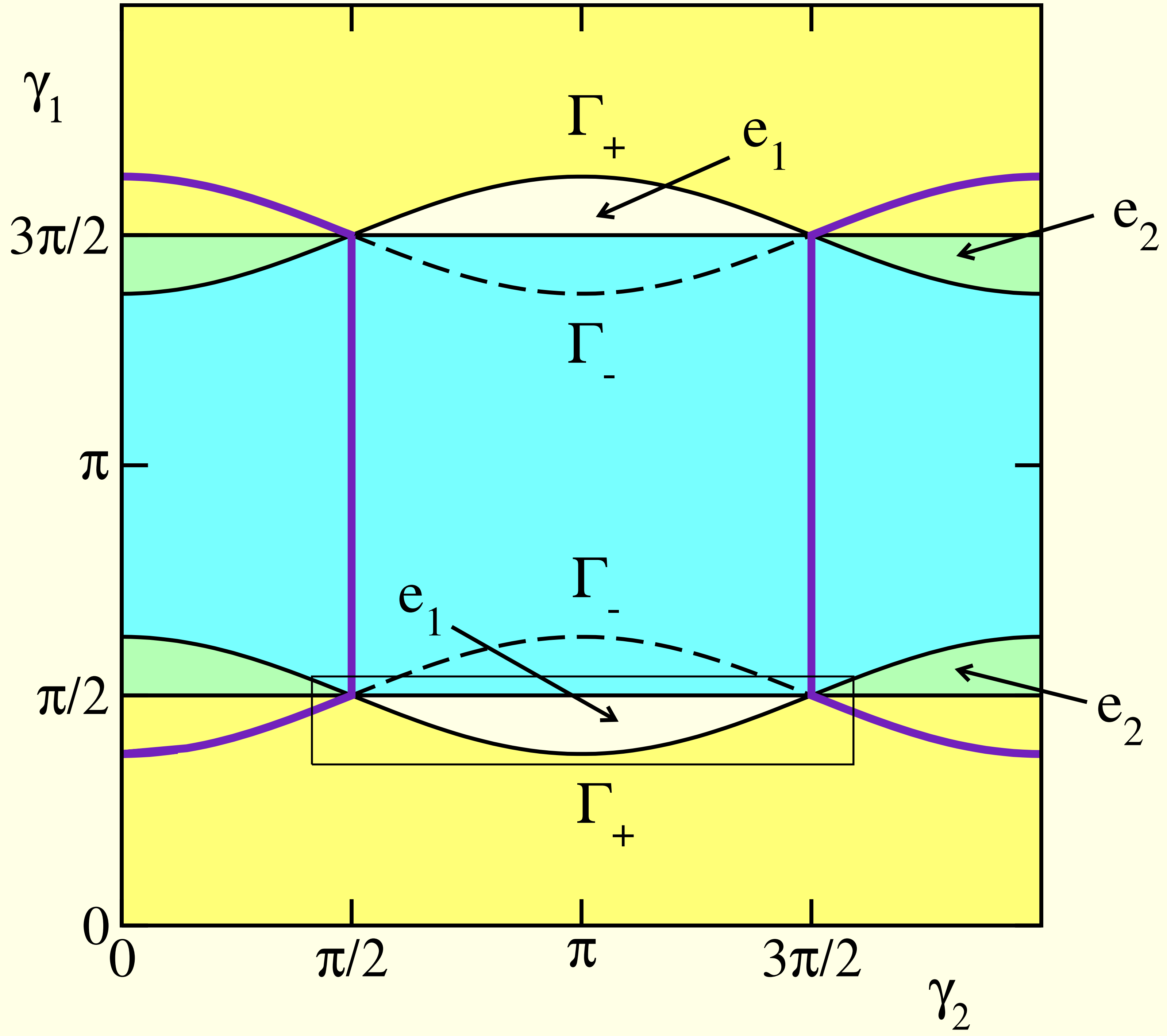
Generalized order parameters $R_m e^{i\Theta_m} = N^{-1} \sum_j e^{im\varphi_j}$

1) frequency can be removed by a transformation to a co-rotating frame

2) coupling strength can be removed by rescaling of time

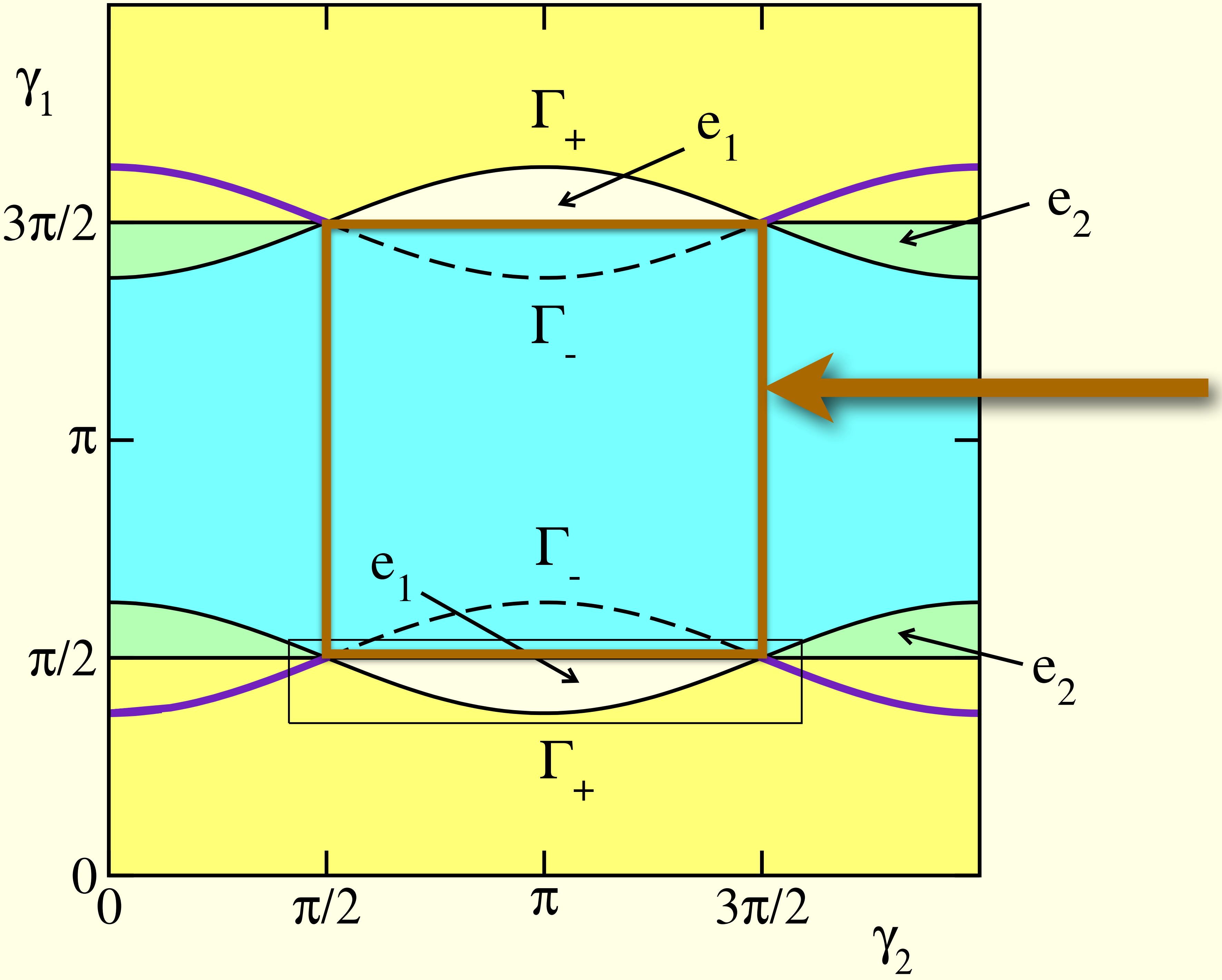
3) parameter ***a=0.2*** is fixed, parameters $\gamma_{1,2}$ are varied

The biharmonic model: stability analysis



P. Clusella, A. Politi, M. Rosenblum, New J. Physics 18 (2016) 093037

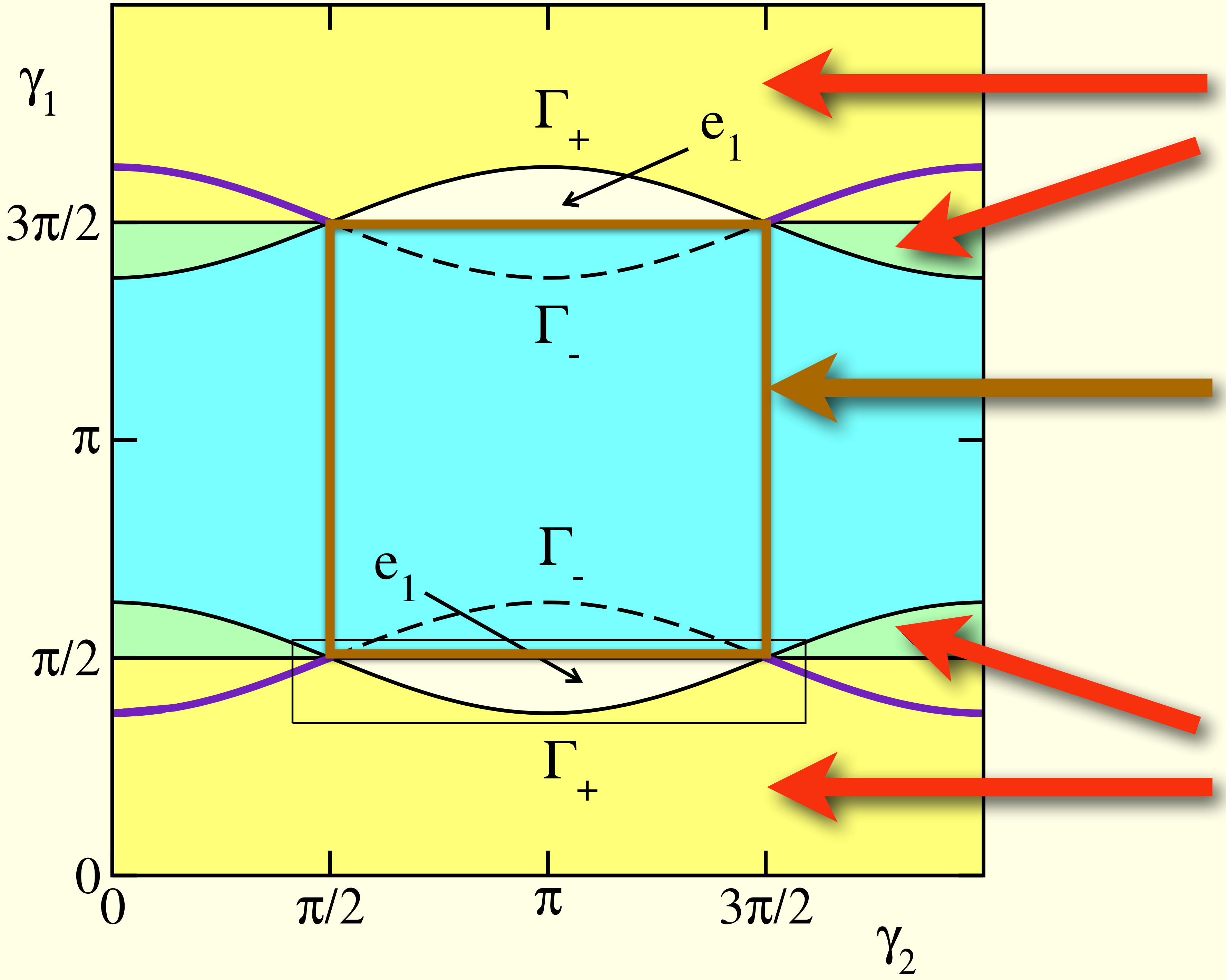
The biharmonic model: stability analysis



Splay state stable here

P. Clusella, A. Politi, M. Rosenblum, New J. Physics 18 (2016) 093037

The biharmonic model: stability analysis



Full synchrony

stable here

Splay state

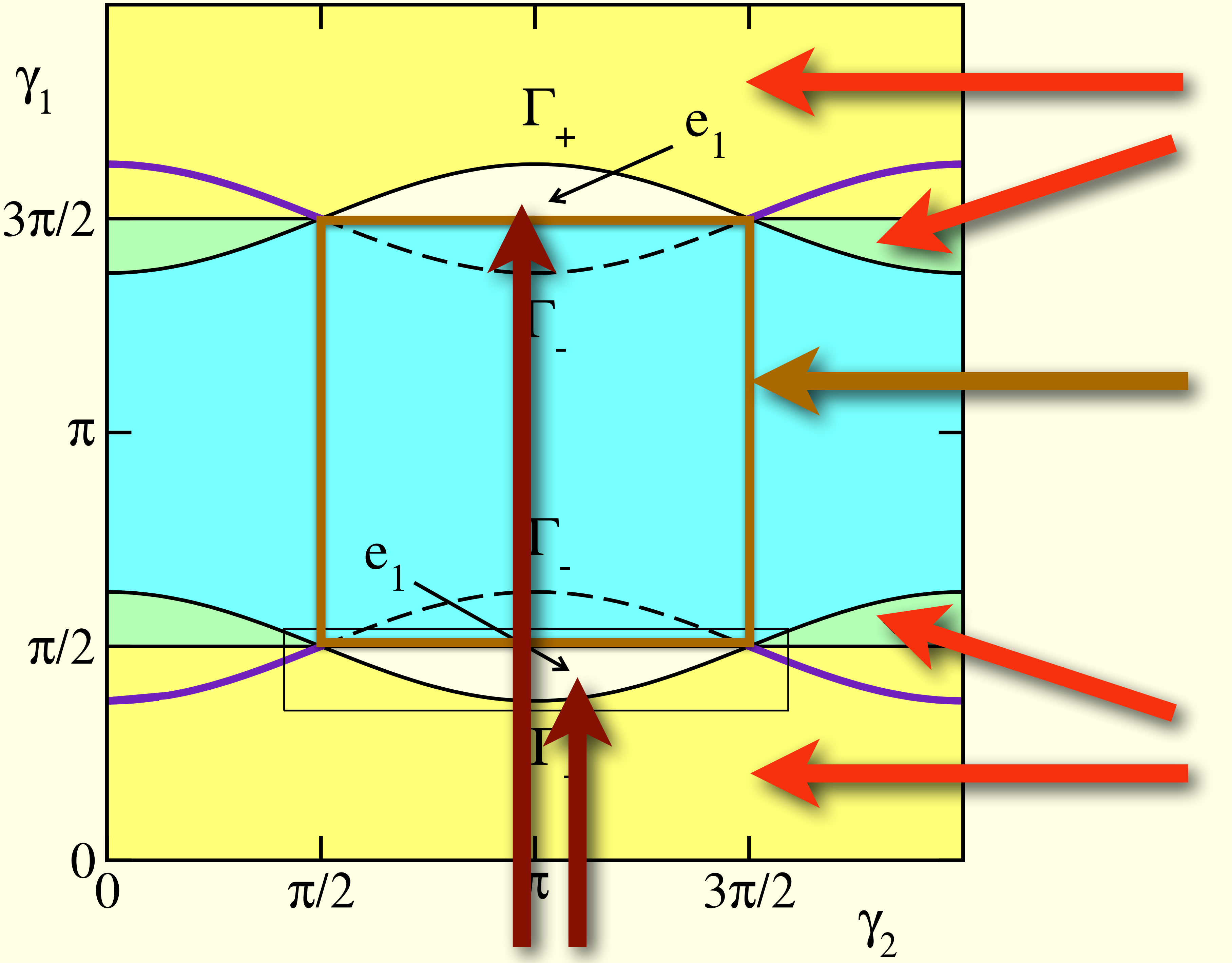
stable here

Full synchrony

stable here

P. Clusella, A. Politi, M. Rosenblum, New J. Physics 18 (2016) 093037

The biharmonic model: stability analysis



Full synchrony

stable here

Splay state

stable here

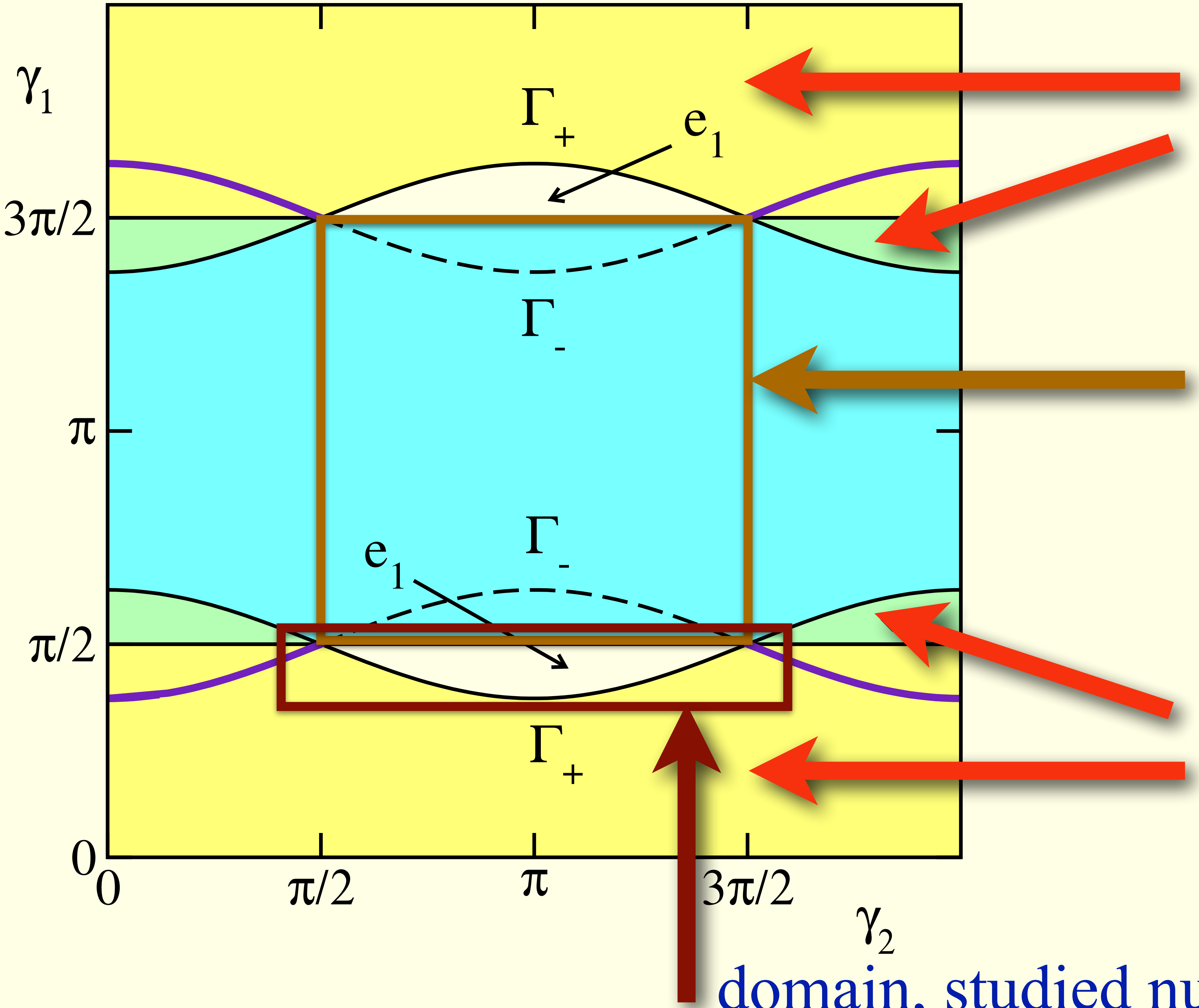
Full synchrony

stable here

synchrony AND splay unstable here

P. Clusella, A. Politi, M. Rosenblum, New J. Physics 18 (2016) 093037

The biharmonic model: stability analysis



Full synchrony

stable here

Splay state

stable here

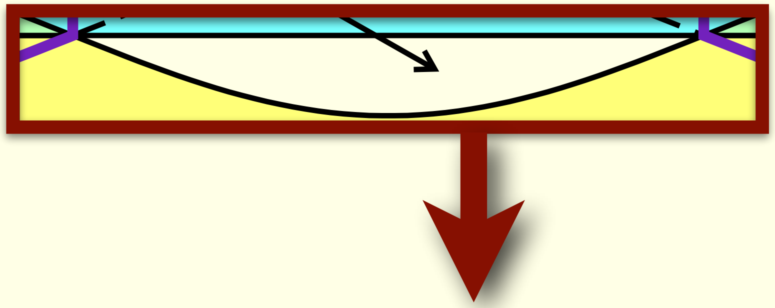
Full synchrony

stable here

domain, studied numerically

P. Clusella, A. Politi, M. Rosenblum, New J. Physics 18 (2016) 093037

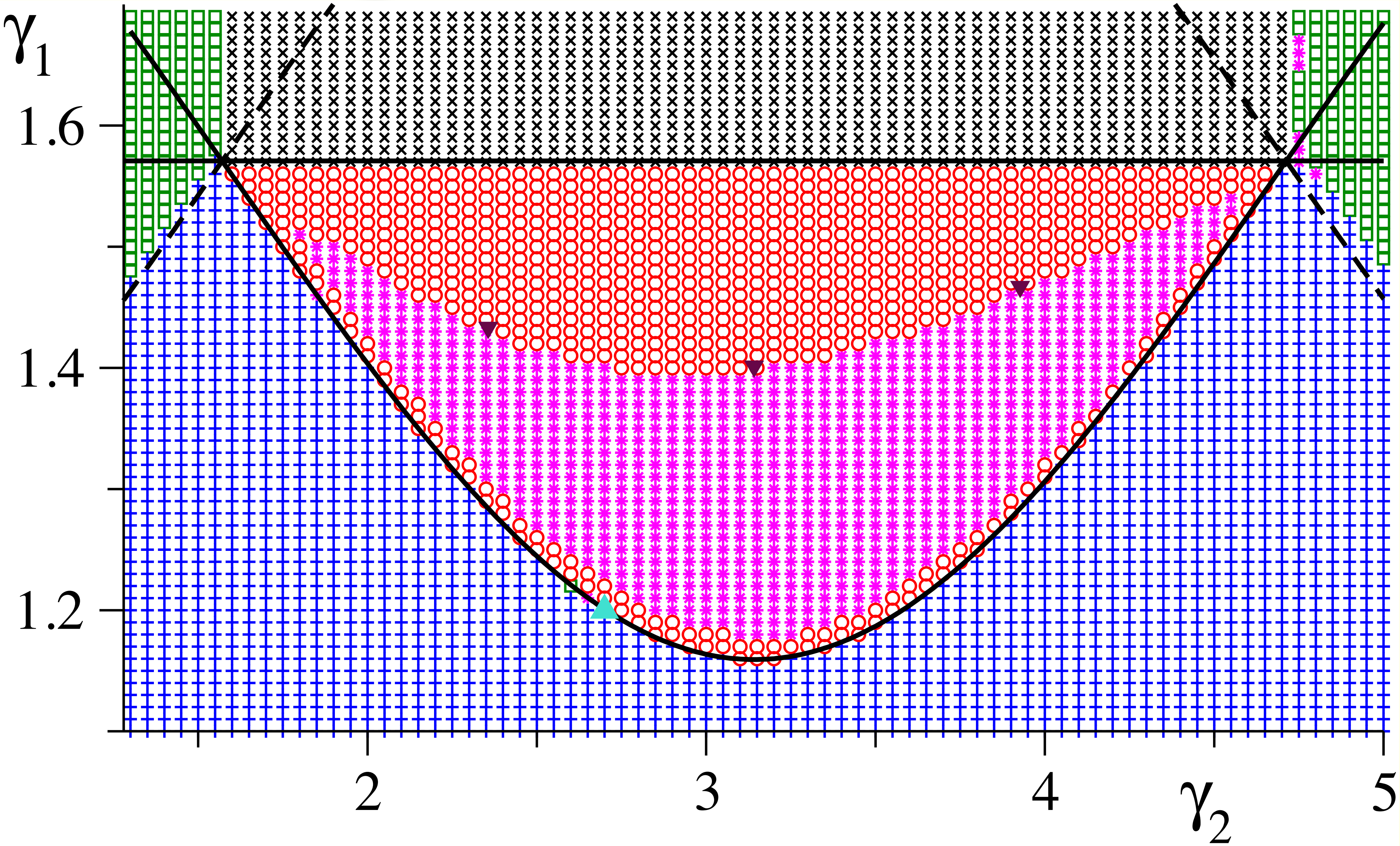
The biharmonic model: numerics



domain, studied numerically

× splay states

□ two clusters

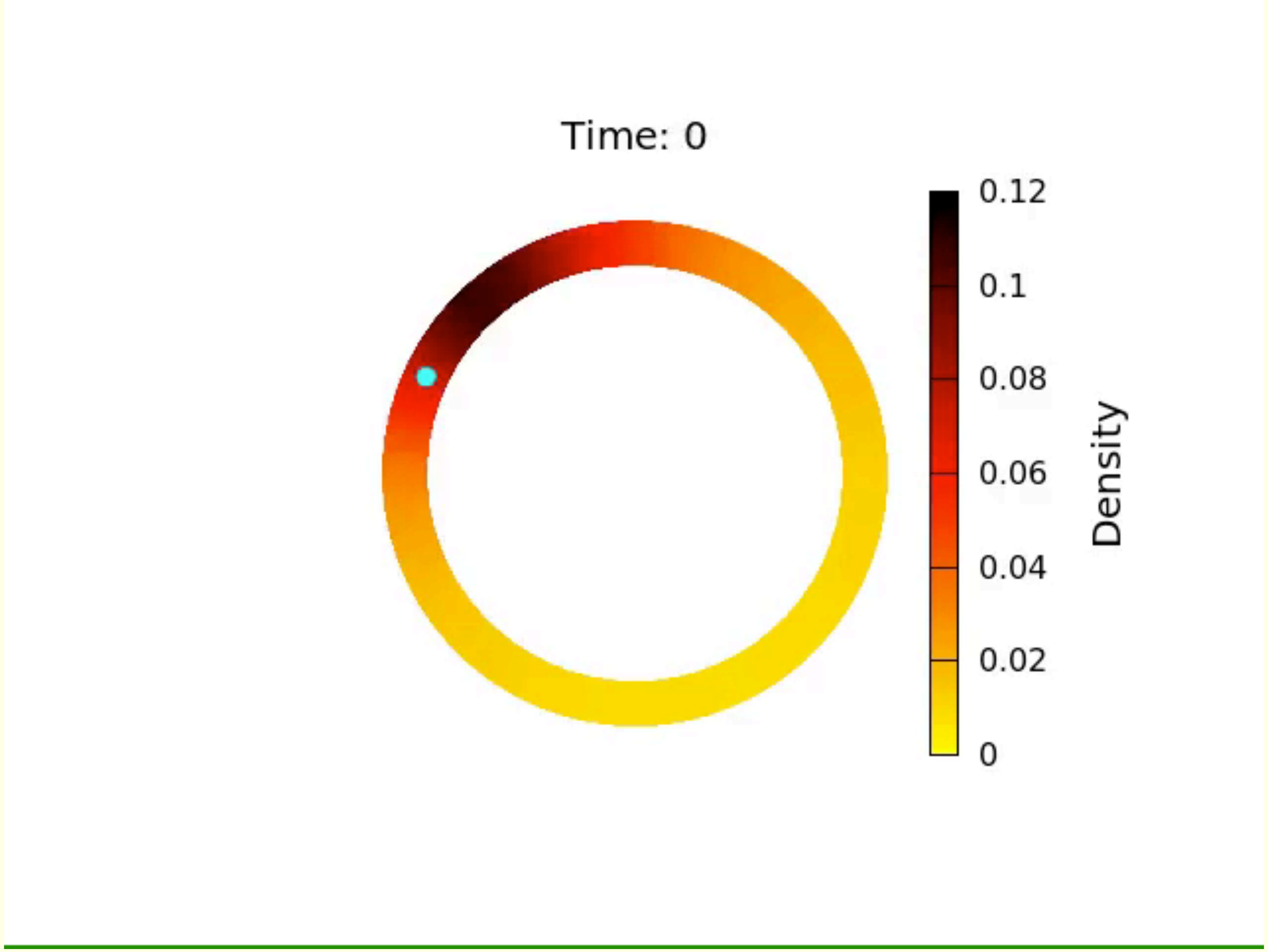


* heteroclinic cycles

\+ synchrony

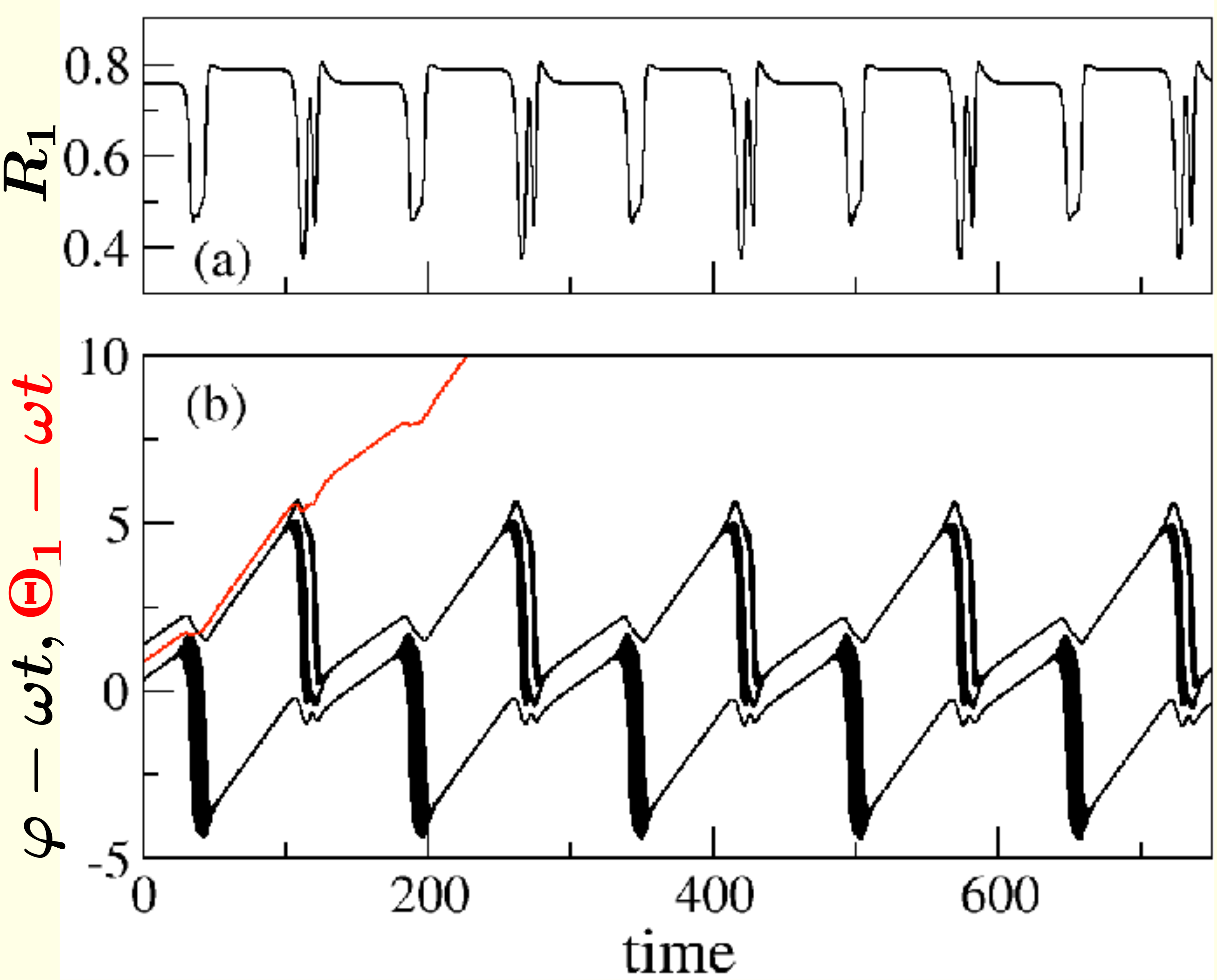
○ partial synchrony

Partial synchrony



Heteroclinic cycles as partially synchronous states

HC in biharmonic model: Hansel et al, 1993; Kori and Kuramoto, 2001



The biharmonic model: numerics

domain, studied numerically

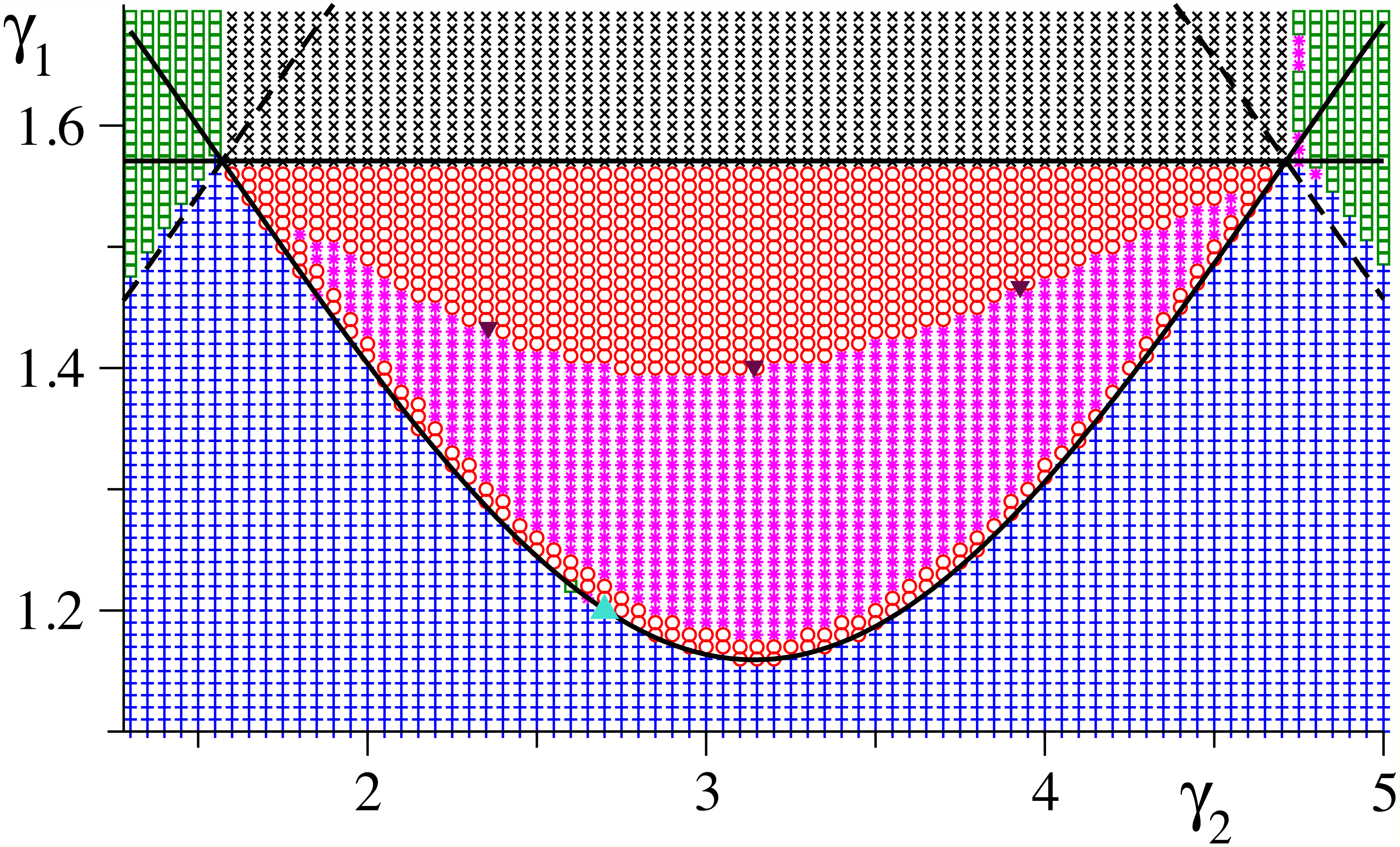
× splay states

□ two clusters

∗ heteroclinic cycles

+ synchrony

○ partial synchrony

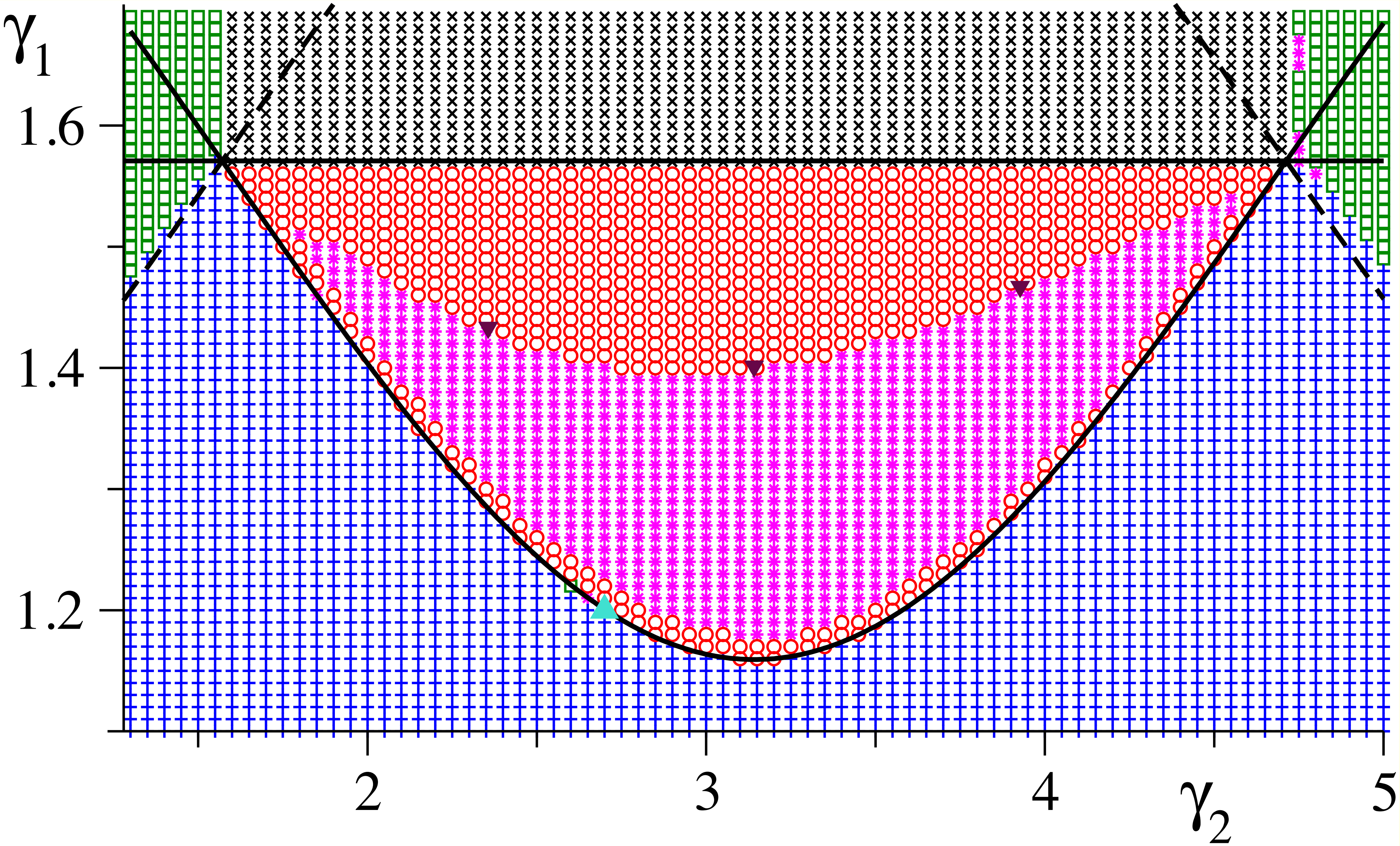


The biharmonic model: numerics

domain, studied numerically

× splay states

□ two clusters



* heteroclinic cycles

\+ synchrony

○ partial synchrony

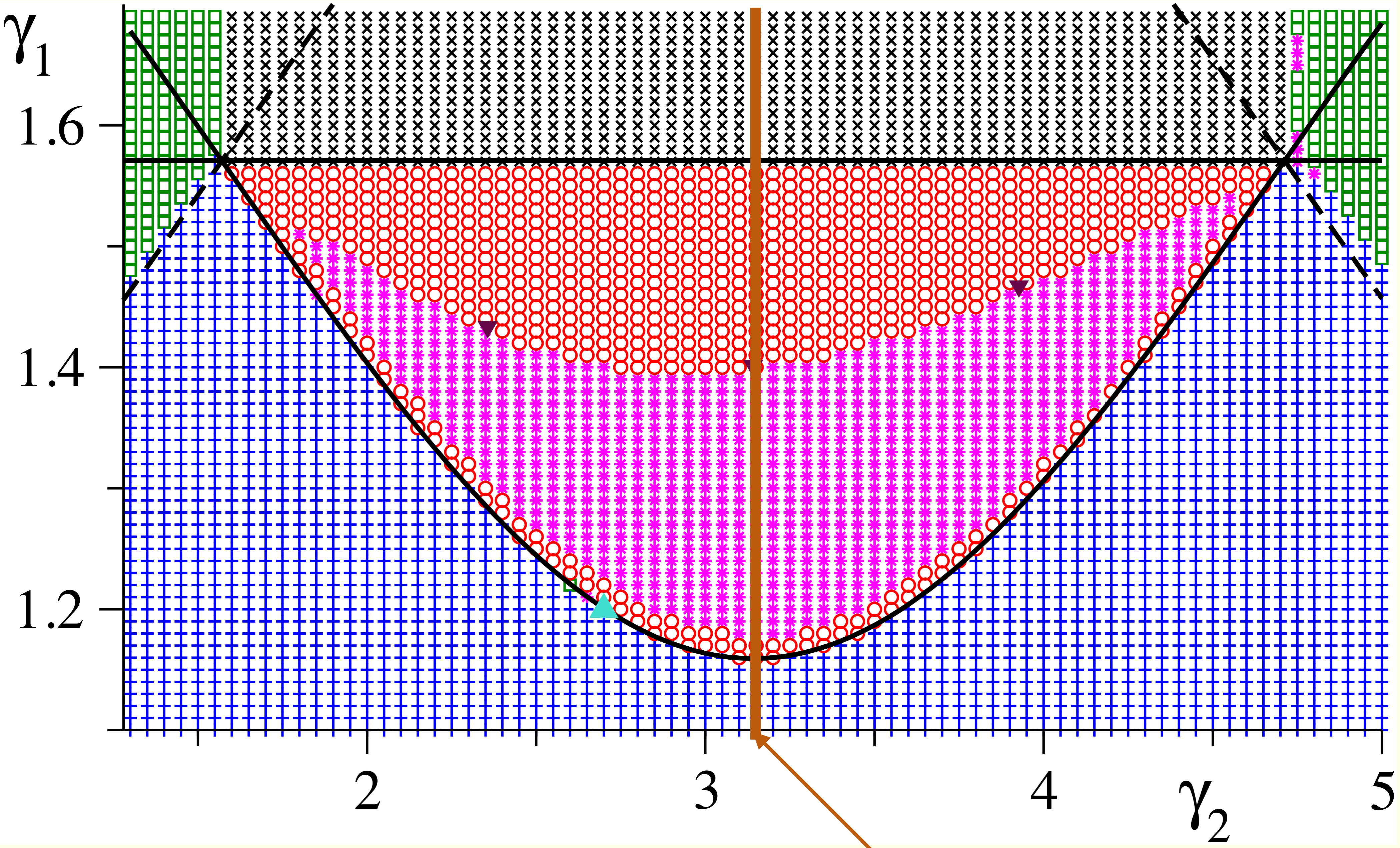
Initial conditions: perturbed splay state!

The biharmonic model: numerics

domain, studied numerically

× splay states

□ two clusters



∗ heteroclinic cycles

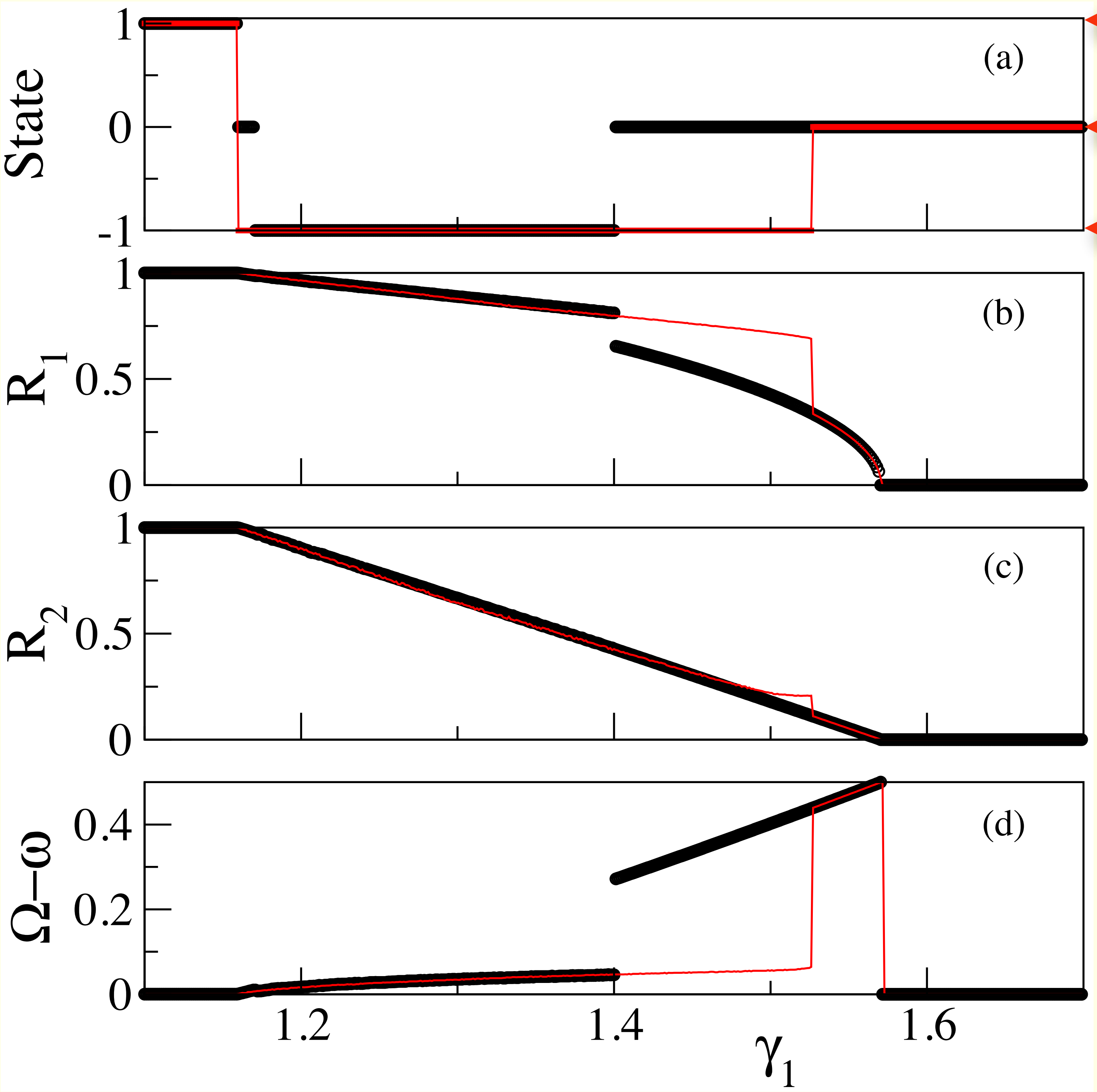
+ synchrony

○ partial synchrony

we go along this line $\gamma_2 = \pi$

Different initial conditions!

The biharmonic model: multistability



synchrony

partial synchrony

heteroclinic cycle

red lines: slow increase of γ_1

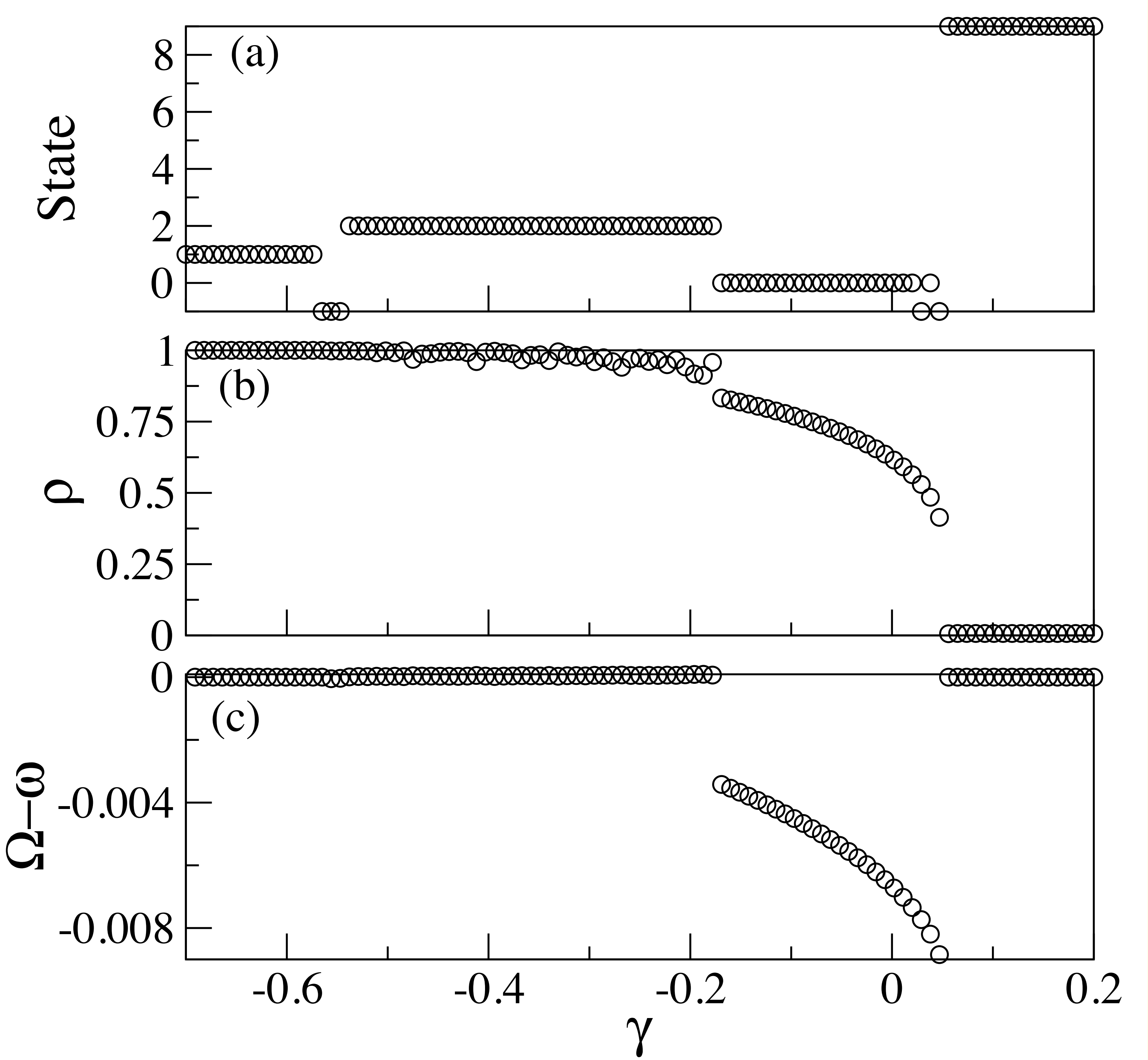
Black symbols:

perturbed splay initial conditions and slow decrease of γ_1

Rayleigh oscillators

$$\ddot{x}_k - \xi(1 - \dot{x}_k^2)\dot{x}_k + x_k = \varepsilon \mathrm{Re}\left[e^{i\gamma}(X + iY)\right]$$

mean fields $X = N^{-1}\sum_k x_k, \quad Y = N^{-1}\sum_k \dot{x}_k$



State = number of clusters

(State = -1: intermediate, unclassified states)

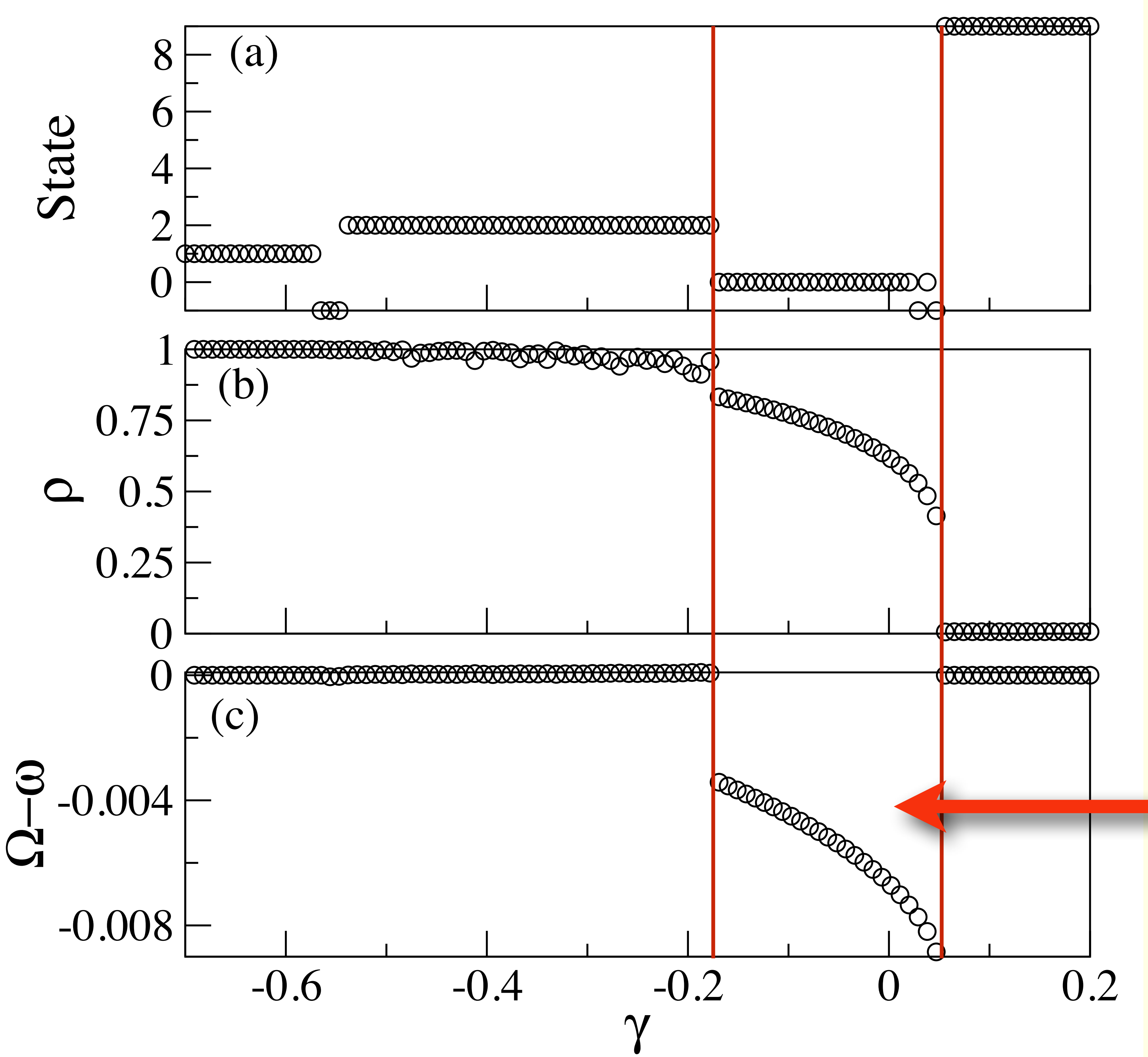
Order parameter

$\rho = \mathrm{rms}(X)/\mathrm{rms}(x)$

Rayleigh oscillators

$$\ddot{x}_k - \xi(1-\dot{x}_k^2)\dot{x}_k + x_k = \varepsilon \mathrm{Re}\left[e^{i\gamma}(X+iY)\right]$$

mean fields $X = N^{-1}\sum_k x_k, \quad Y = N^{-1}\sum_k \dot{x}_k$



State = number of clusters

(State = -1: intermediate, unclassified states)

Order parameter

$\rho = \mathrm{rms}(X)/\mathrm{rms}(x)$

quasiperiodic partial synchrony

General case:
When do we expect complex solutions?

1) tendency to synchrony is not monotonic and/or

2) **both** splay state and synchrony are unstable

The system settles at some intermediate state

We expect: clusters
chimeras
quasiperiodic partially synchronous states

General case:
When do we expect complex solutions?

1) **tendency to synchrony is not monotonic and/or**

2) **both** splay state and synchrony are unstable

The system settles at some intermediate state

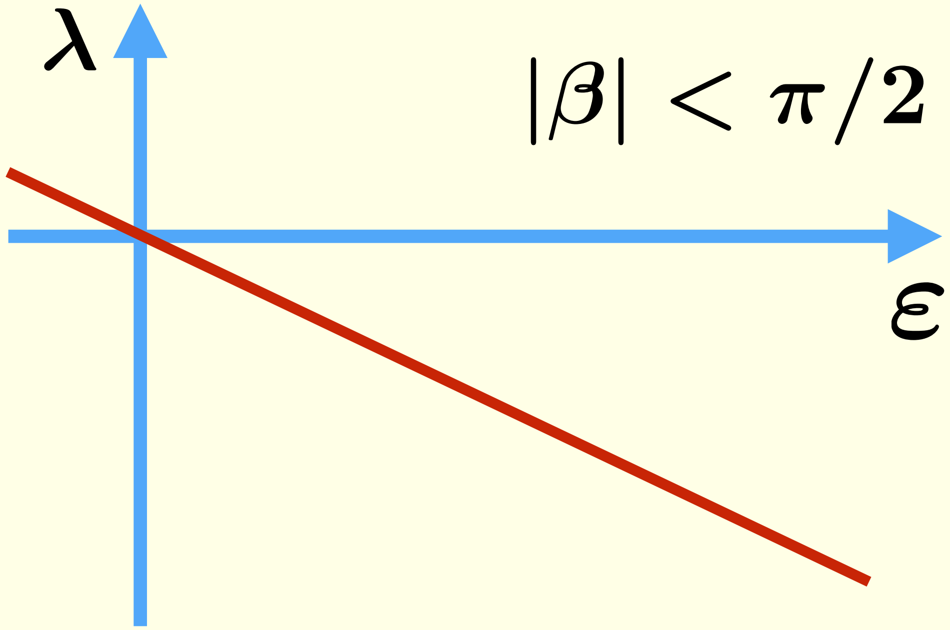
We expect: clusters
chimeras
quasiperiodic partially synchronous states

Stability of the synchronous state

The Kuramoto-Sakaguchi model, identical oscillators:

Synchronous (one-cluster) state is stable, if $\boldsymbol{\lambda = -\varepsilon \cos\beta < 0}$

eigenvalue



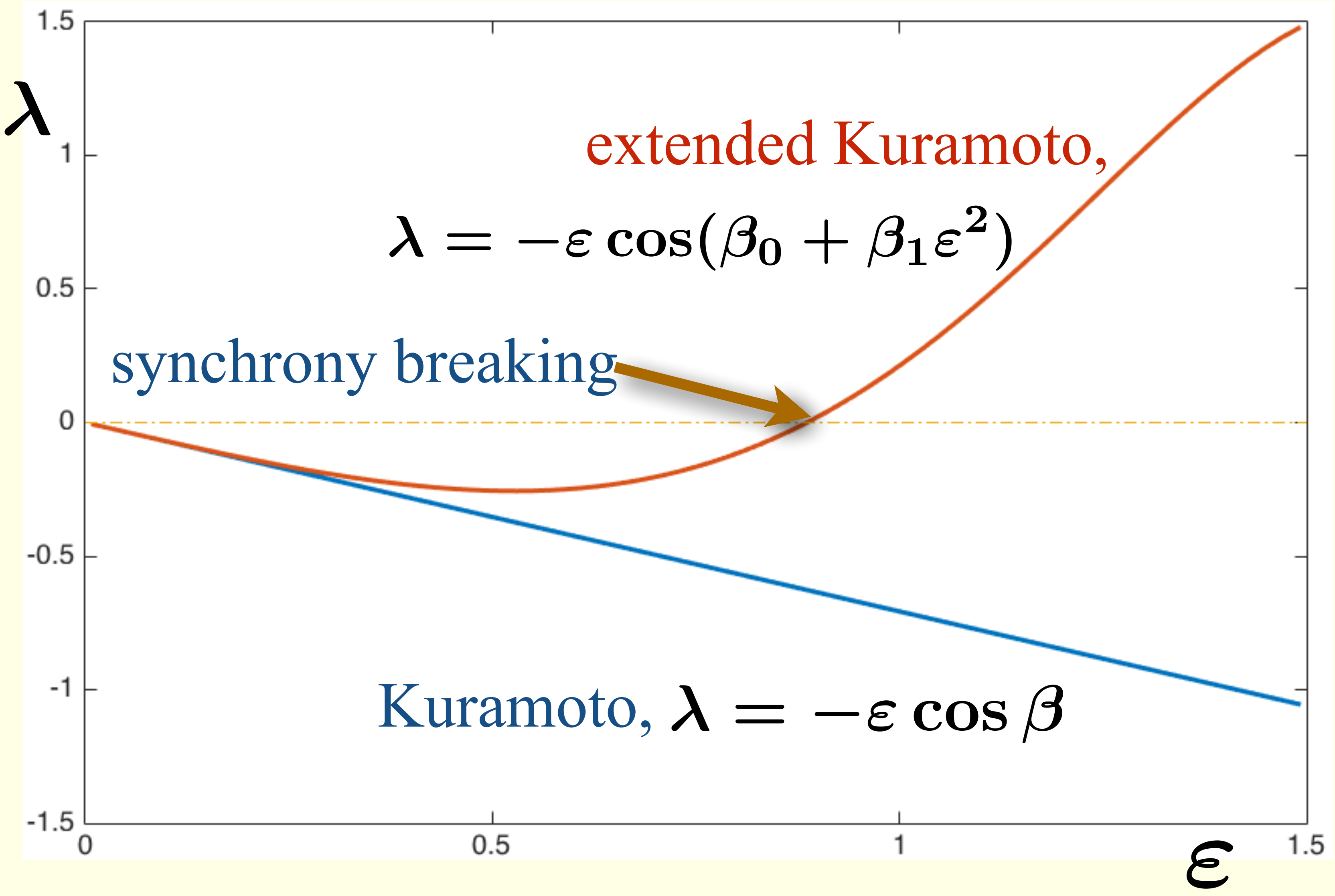
For this model: stability is proportional to coupling

==> tendency to synchrony increases with ε

Linear vs nonlinear coupling

Extended Kuramoto-Sakagichi model (particular case):

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)$$



nonlinear coupling

linear coupling

Partial synchrony and quasiperiodic dynamics after synchrony breaking

A solvable model for Quasiperiodic Partial Synchrony

Nonlinearly coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega_0)a_k - (1 + i\kappa)|a_k|^2 a_k$$
$$+(\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \ ,$$

complex mean field:

$$A = N^{-1}\sum_j a_j$$

linear and nonlinear

mean field coupling

The solvable model: phase approximation

Nonlinearly coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega_0)a_k - (1 + i\kappa)|a_k|^2 a_k$$
$$+(\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \ ,$$

complex mean field:

$$A = N^{-1}\sum_j a_j$$

linear and nonlinear

mean field coupling

Phase approximation: nonlinear Kuramoto-Sakagichi model

$$\boxed{\dot{\varphi}_k = \omega + \mathcal{E}(R;\varepsilon_{1,2},\eta_{1,2})R\sin[\Theta - \varphi_k + \beta(R;\varepsilon_{1,2},\eta_{1,2})]}$$

A solvable particular case:

$$\boxed{\dot{\varphi}_k = \omega + \varepsilon R\sin(\Theta - \varphi_k + \beta_0 + \beta_1\varepsilon^2 R^2)}$$

Nonlinear coupling: another setup

Stuart-Landau oscillators, coupled via a common nonlinear medium

$$\dot{a}_k = (\mu + i\omega_k)a_k - |a_k|^2 a_k + e^{i\beta}\mathcal{F}$$

$$\dot{\mathcal{F}} = -\gamma\mathcal{F} + i\nu\mathcal{F} + i\eta|\mathcal{F}|^2\mathcal{F} + \tilde{\varepsilon}A$$

Phase approximation yields same phase model:

$$\boxed{\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)}$$

Nonlinear coupling: numerics

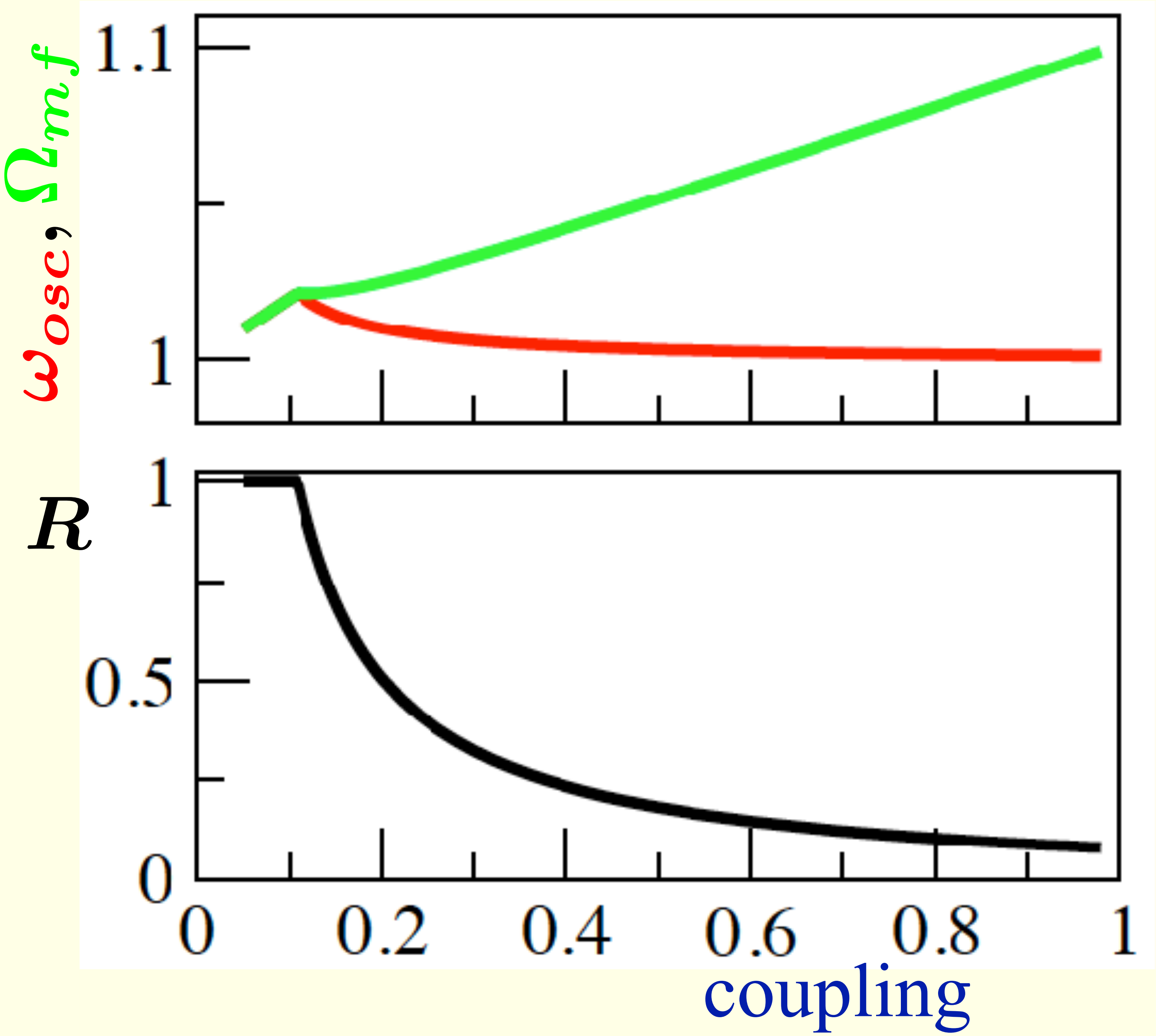
Stuart-Landau oscillators, coupled via a common nonlinear medium

$$\dot{a}_k = (\mu + i\omega_k)a_k - |a_k|^2 a_k + e^{i\beta}\mathcal{F}$$

$$\dot{\mathcal{F}} = -\gamma\mathcal{F} + i\nu\mathcal{F} + i\eta|\mathcal{F}|^2\mathcal{F} + \tilde{\varepsilon}A$$

Good agreement:

- full system
- phase model
- theory



Parameters:

$\gamma = 0.5$

$\eta = 10^3$

$\beta = 0.475\pi$

Qualitative discussion: order parameter

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)$$

Let $|\beta_0| < \pi/2$ $\longrightarrow$ asynchrony ($R = 0$) is unstable

Synchrony ($R = 1$) is stable if $\beta_0 + \beta_1 \varepsilon^2 < \pi/2$

unstable if $\beta_0 + \beta_1 \varepsilon^2 > \pi/2$

Hence, for $\varepsilon > \varepsilon_{crit} = \sqrt{(\pi/2 - \beta_0)/\beta_1}$

the system settles **between asynchrony and synchrony**

Theory: $\boxed{\varepsilon > \varepsilon_{crit} : \quad \beta(R, \varepsilon) = \beta_0 + \beta_1 \varepsilon^2 R^2 = \pi/2}$

$$\boxed{R = \varepsilon_{crit}/\varepsilon}$$

Self-organized partial synchrony

Qualitative discussion: frequency difference

In the partially synchronous state $R < 1$

Watanabe-Strogatz theory: cluster states are not possible

Hence, all phases are different

Hence, instantaneous frequencies

$$\dot{\varphi}_k = \omega + \varepsilon R \sin(\Theta - \varphi_k + \beta_0 + \beta_1 \varepsilon^2 R^2)$$

are all different as well

We denote: $\langle \dot{\phi} \rangle = \Omega \,, \langle \dot{\Theta} \rangle = \nu$

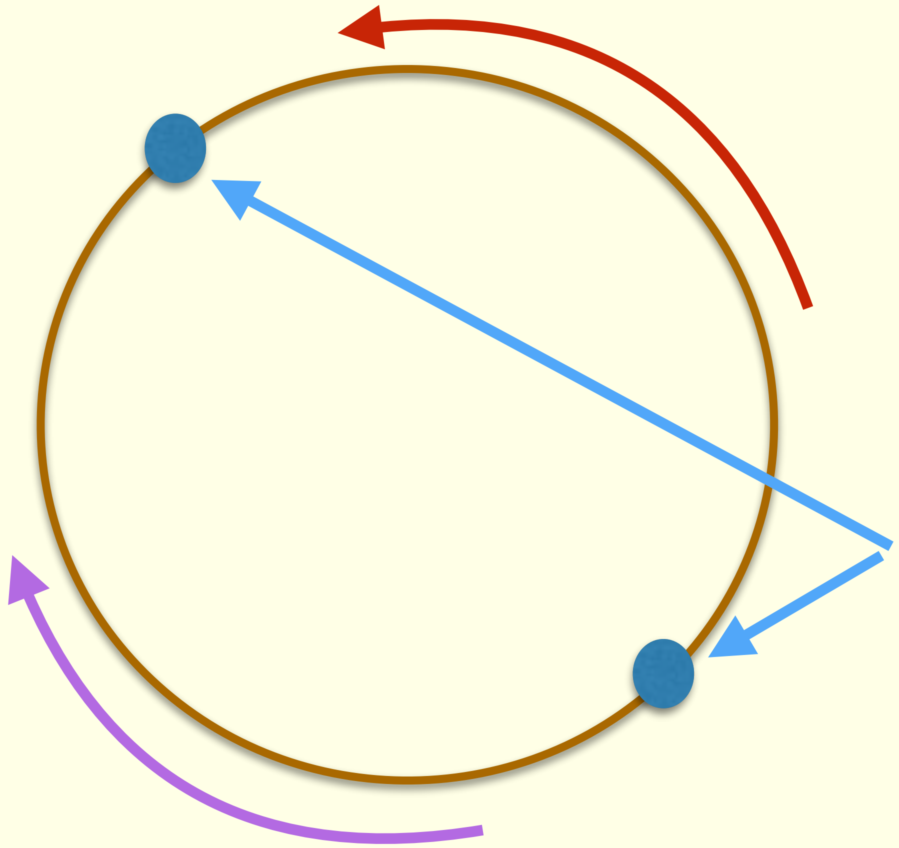
oscillator frequency

mean field frequency

We argue that $\Omega \neq \nu$

Qualitative discussion: frequency difference II

Suppose the contrary, $\Omega = \nu$, and consider the motion in the frame, rotating with the mean field



Then, depending on their phase, some oscillators are faster than mean field, and some are slower

There must be points where relative velocity is zero $\longrightarrow$ the points should cluster

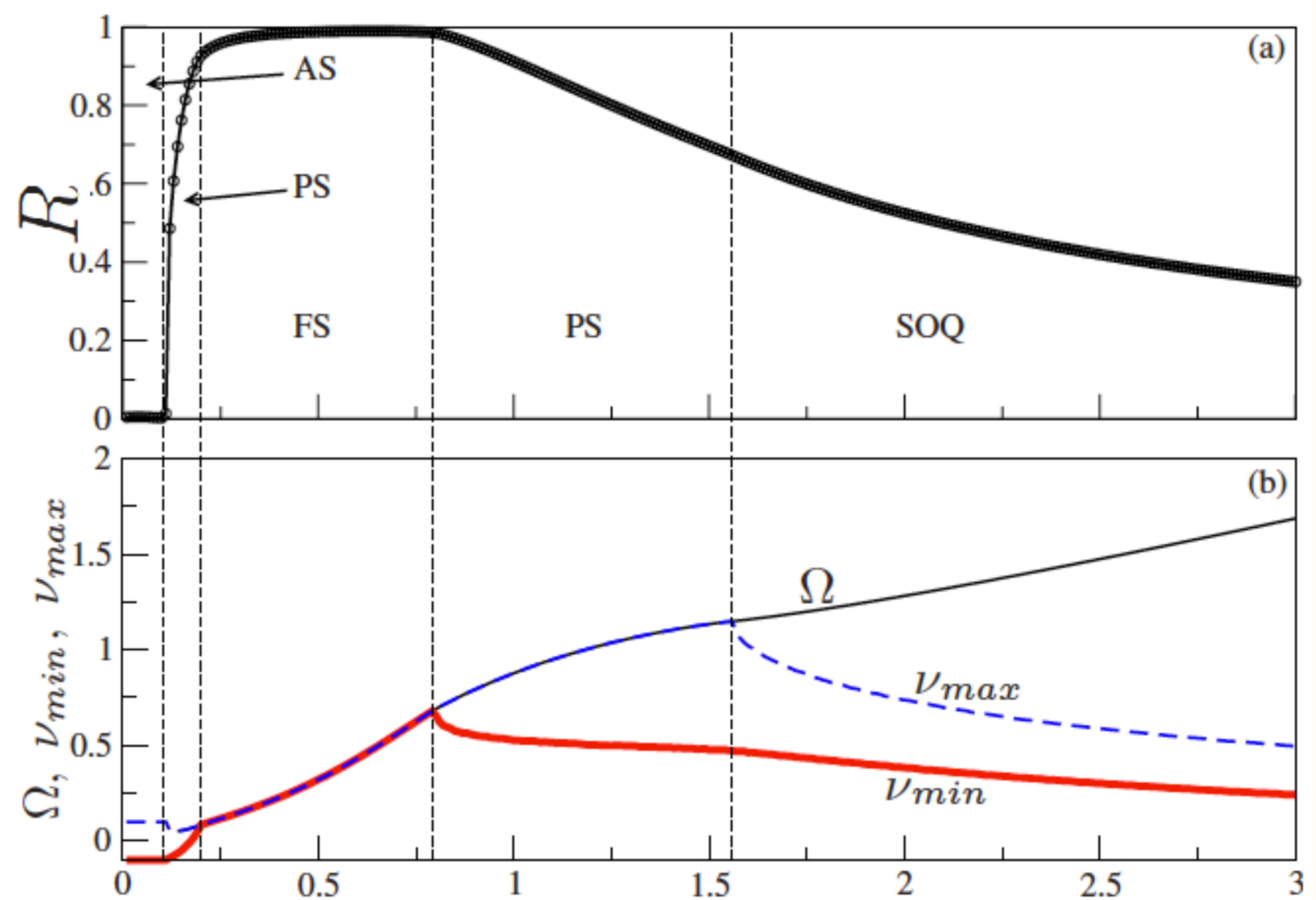
Clusters are not possible, hence oscillators are either always faster, or always slower than the mean field, thus $\boxed{\Omega \neq \nu}$

Theory: $\Omega = \omega + \frac{\varepsilon_{crit}^2}{\varepsilon}$, $\nu = \omega + \frac{\varepsilon^2 + \varepsilon_{crit}^2}{2\varepsilon}$

Quasiperiodic dynamics

Nonidentical oscillators

Theory for uniform frequency distribution

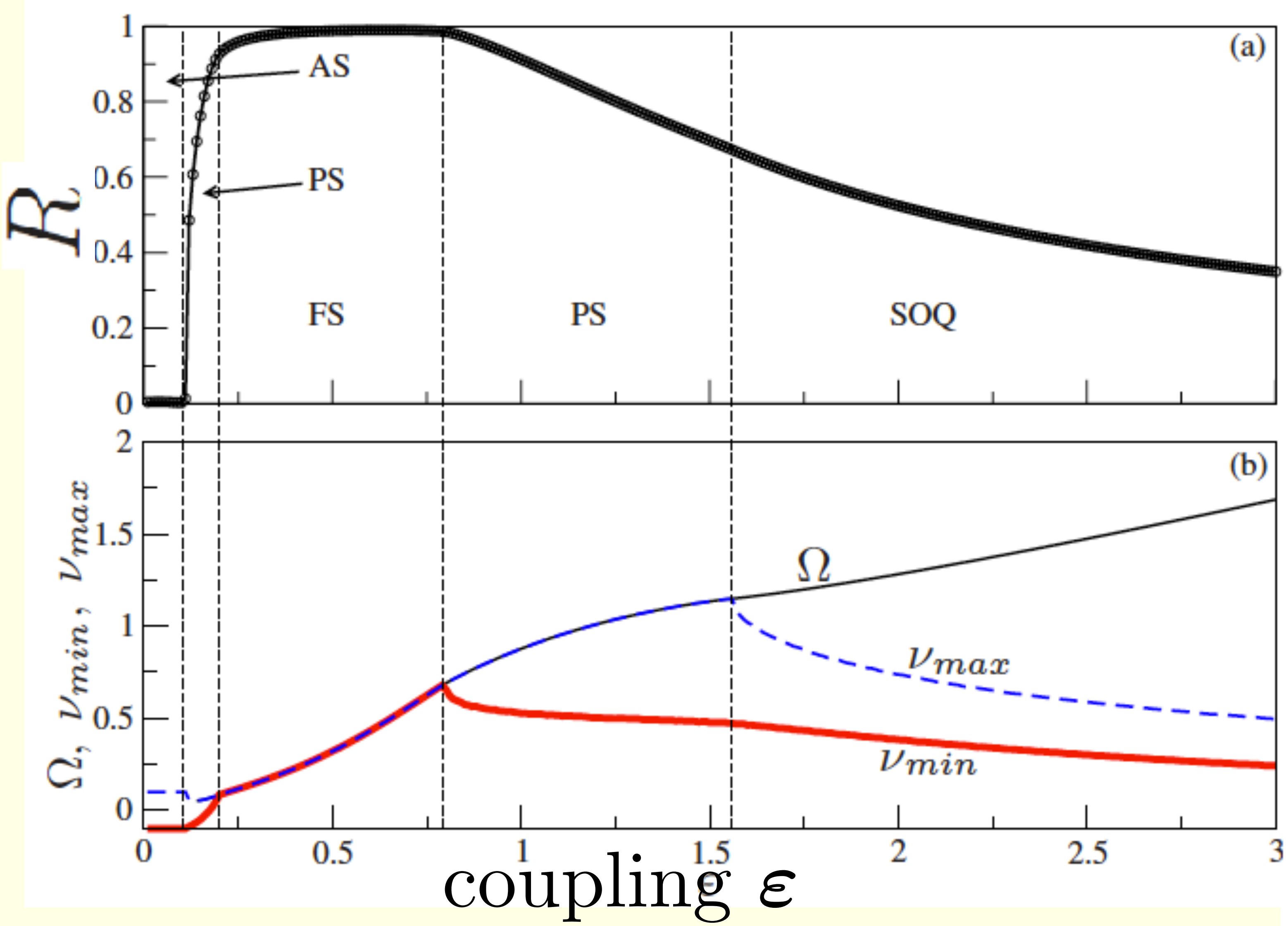


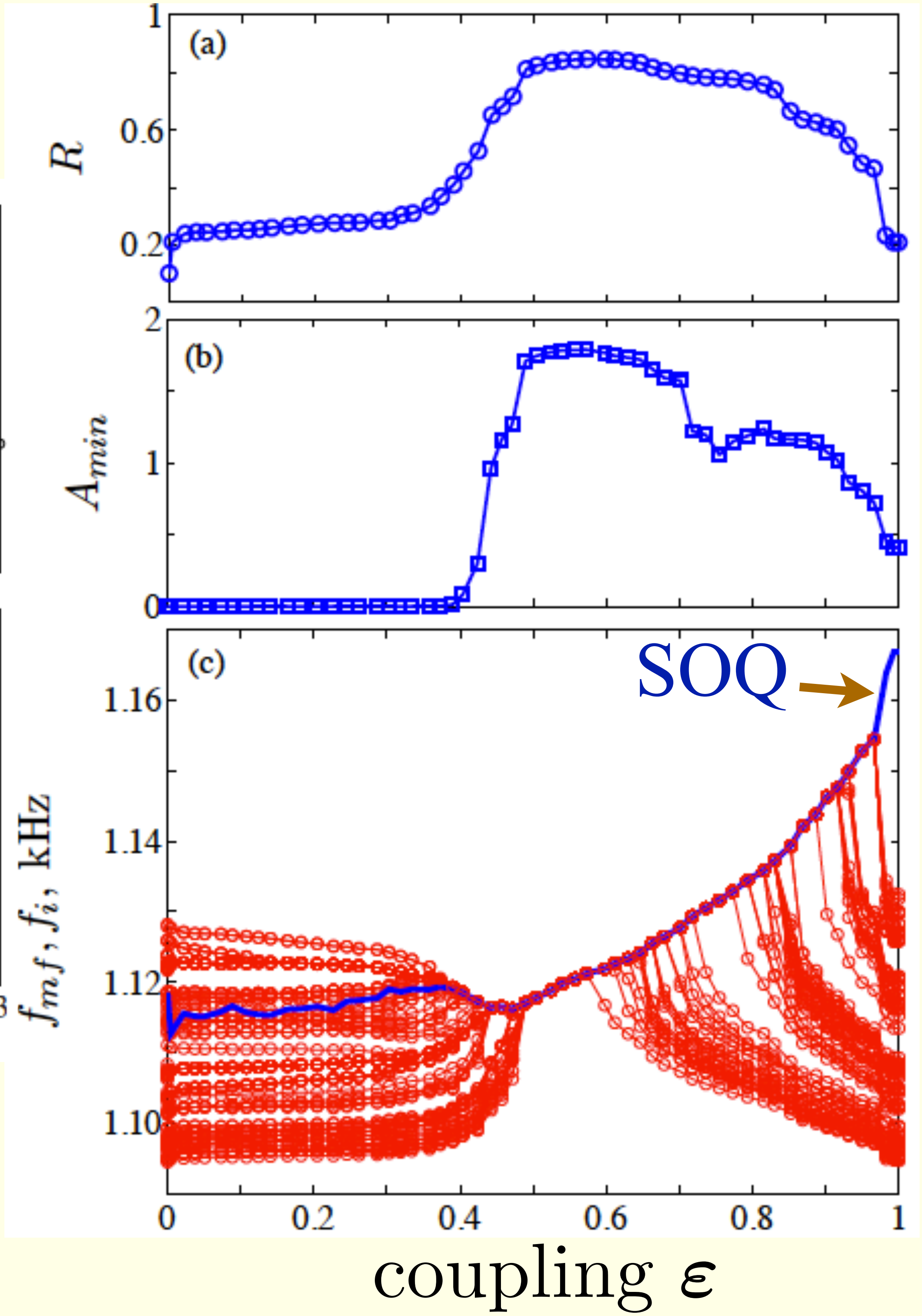
coupling ε

Baibolatov et al., Phys. Rev. E (2010)

Theory vs experiment

Theory for uniform frequency distribution





Baibolatov et al., Phys. Rev. E (2010)

The solvable model: beyond phase approximation

Nonlinearly coupled Stuart-Landau oscillators:

$$\dot{a}_k = (1 + i\omega_0)a_k - (1 + i\kappa)|a_k|^2 a_k$$

$$+(\varepsilon_1 + i\varepsilon_2)A - (\eta_1 + i\eta_2)|A|^2 A \;,$$

complex mean field:

$$A = N^{-1}\sum_j a_j$$

linear and nonlinear

mean field coupling

Stability of the synchronous state

$$a_1 = a_2 = \ldots = a_N = re^{i\varphi} = A \quad \text{with}$$

$$r^2 = \frac{1+\varepsilon_1}{1+\eta_1} \quad \text{and} \quad \dot{\varphi} = \Omega = \omega_0 + \varepsilon_2 - \frac{(\kappa+\eta_2)(1+\varepsilon_1)}{1+\eta_1}$$

The solvable model: beyond phase approximation

Stability of the synchronous state

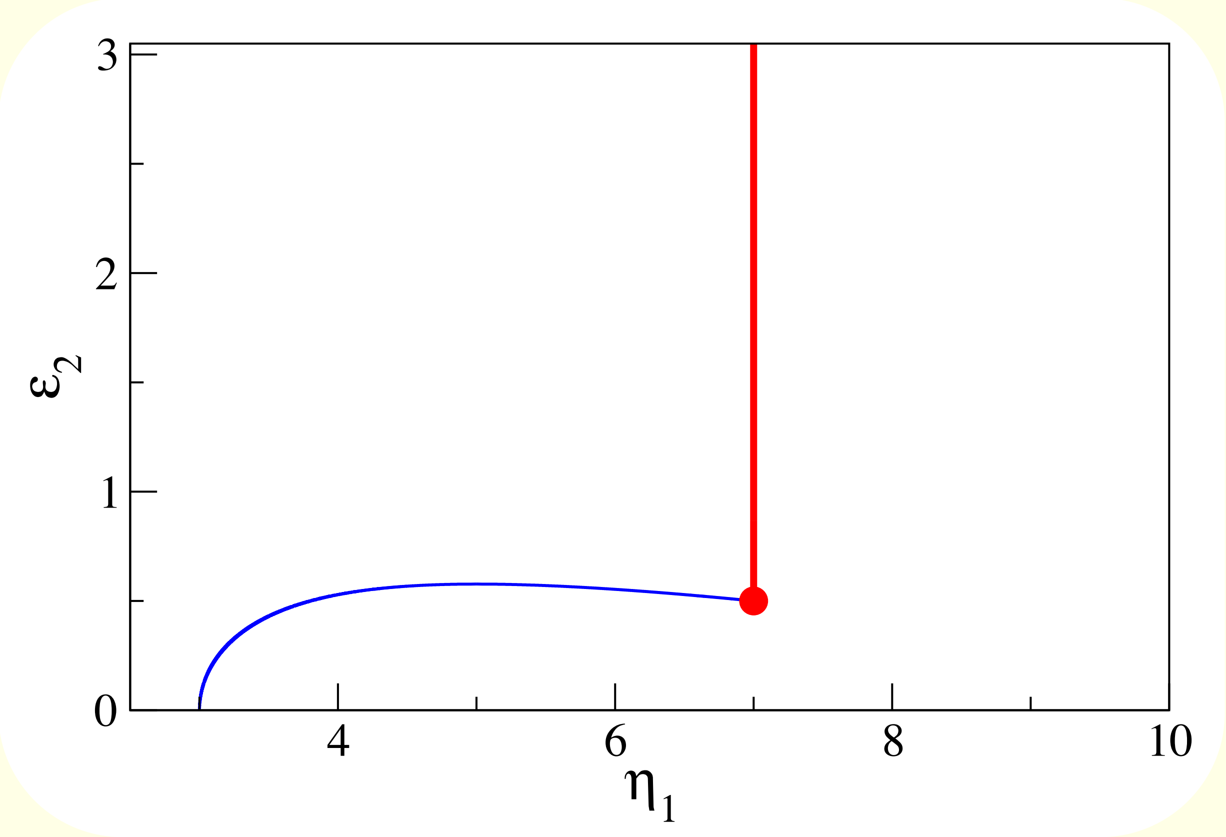
$$a_1 = a_2 = \ldots = a_N = re^{i\varphi} = A$$

Eigenvalues:

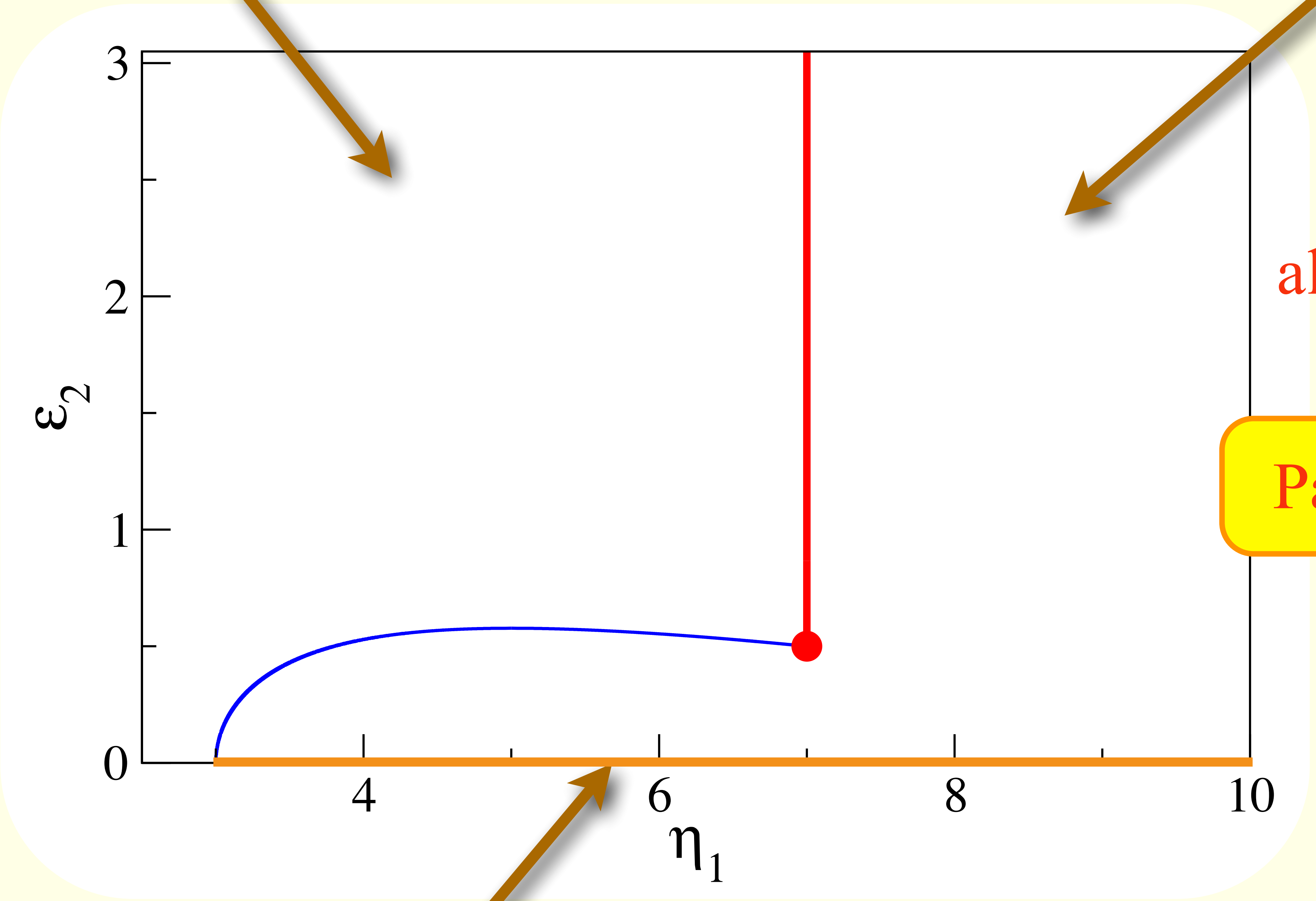
$$\lambda_{1,2} = (1-2r^2) \pm \sqrt{(1-3\kappa^2)r^4 + 4(\omega_0-\Omega)\kappa r^2 - (\omega_0-\Omega)^2}$$

A special case: $\kappa = 0, \eta_2 = 0, \varepsilon_1 = 3, \varepsilon_2 \geq 0$

$$\longrightarrow \quad \lambda_{1,2} = (1-2r^2) \pm \sqrt{r^4 - \varepsilon^2}$$



Stability diagram



Synchrony stable

Synchrony unstable

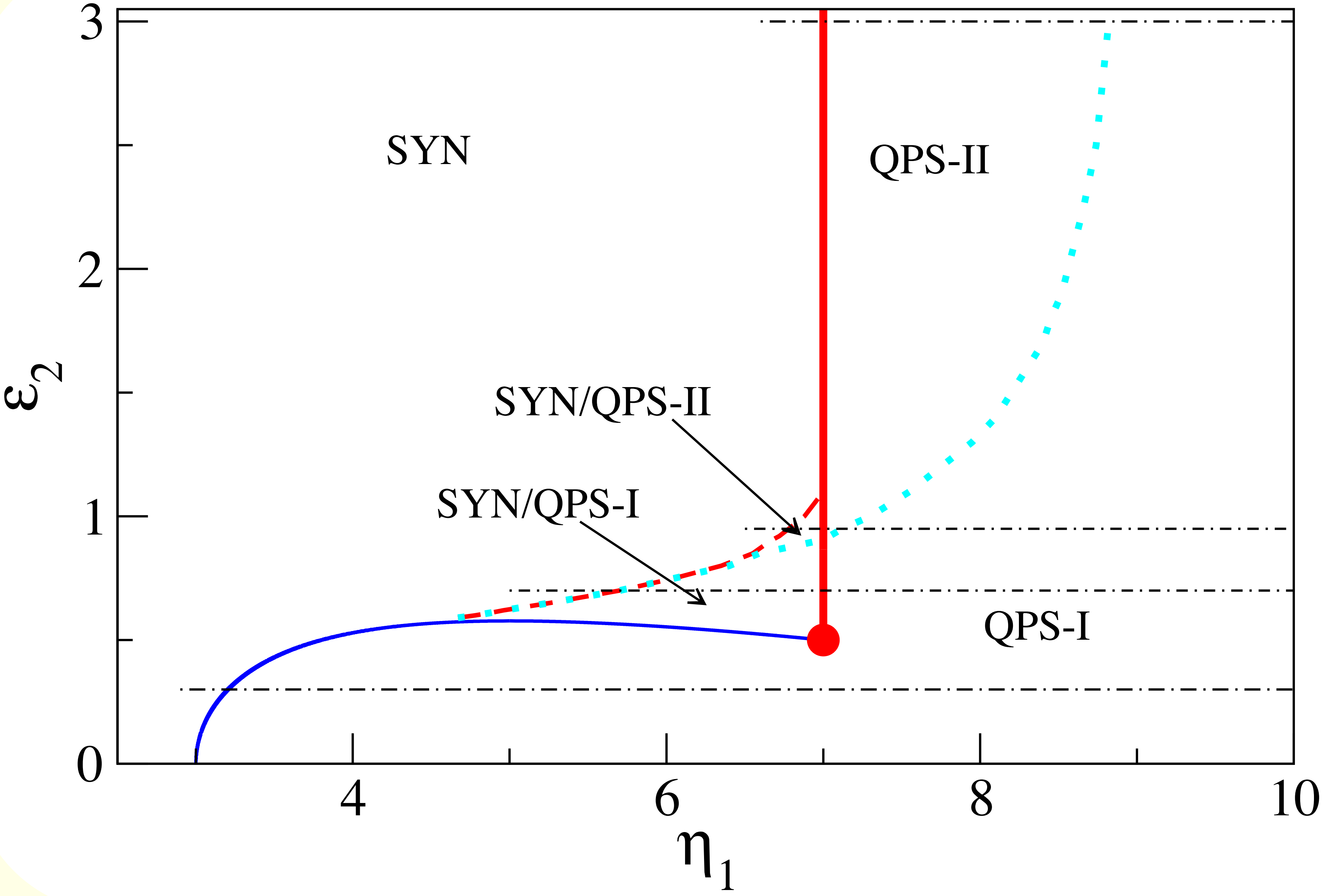
Asynchrony always unstable

Partial synchrony

Neutrally stable bunch state, $r = 1, \Omega = \omega_0, R = \sqrt{\varepsilon_1 / \eta_1}$

Numerics

Mean field frequency ν, oscillators frequency $\Omega = \langle \dot{\varphi} \rangle$



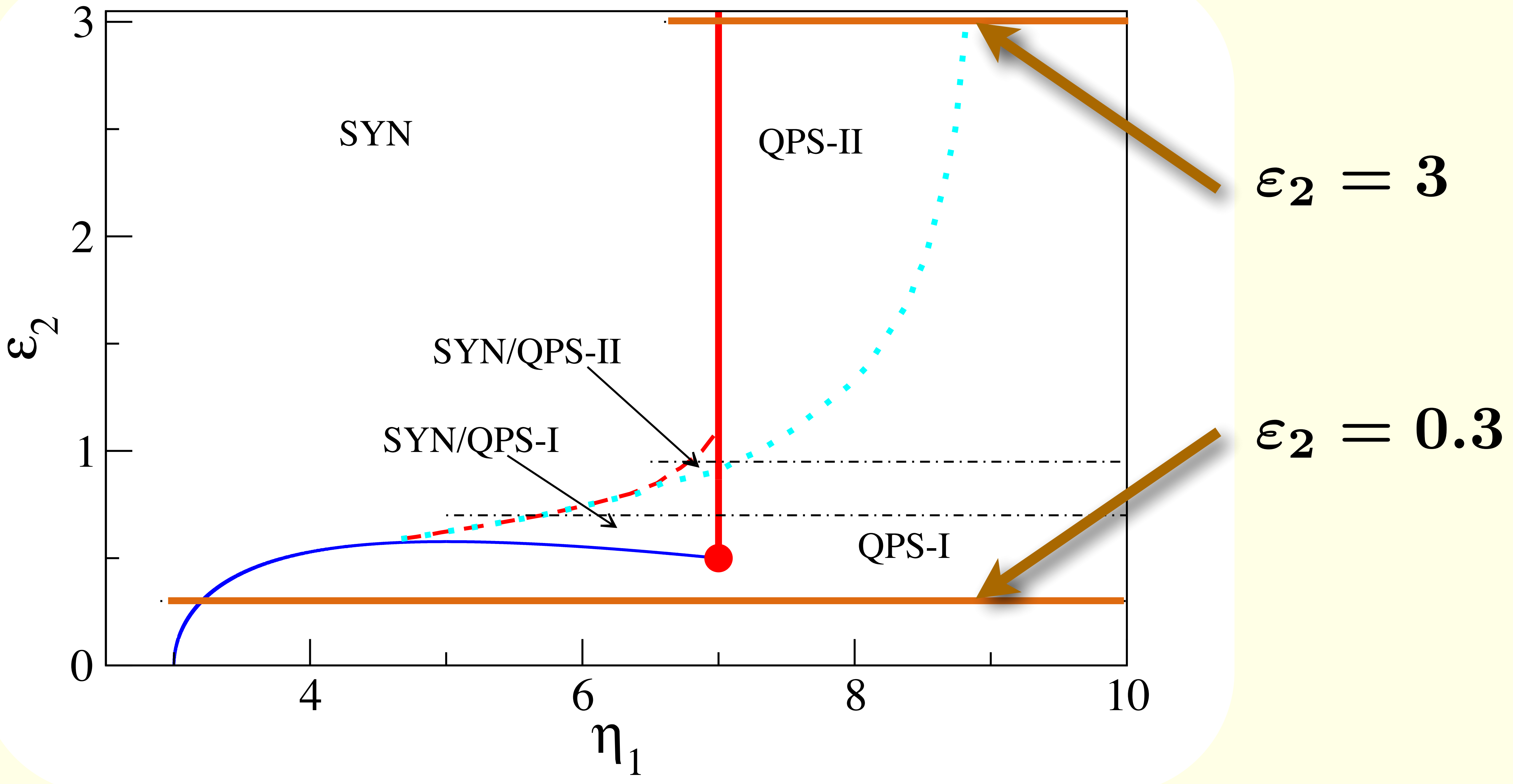
Quasiperiodic partial synchrony type I (QPS-I): $\nu \neq \Omega$

Quasiperiodic partial synchrony type II (QPS-II): $\nu = \Omega$,

quasiperiodicity due to **amplitude modulation**

Numerics

Mean field frequency $\boldsymbol{\nu}$, oscillators frequency $\boldsymbol{\Omega = \langle \dot{\varphi} \rangle}$



Quasiperiodic partial synchrony type I (QPS-I): $\boldsymbol{\nu \neq \Omega}$

Quasiperiodic partial synchrony type II (QPS-II): $\boldsymbol{\nu = \Omega}$,

quasiperiodicity due to **amplitude modulation**

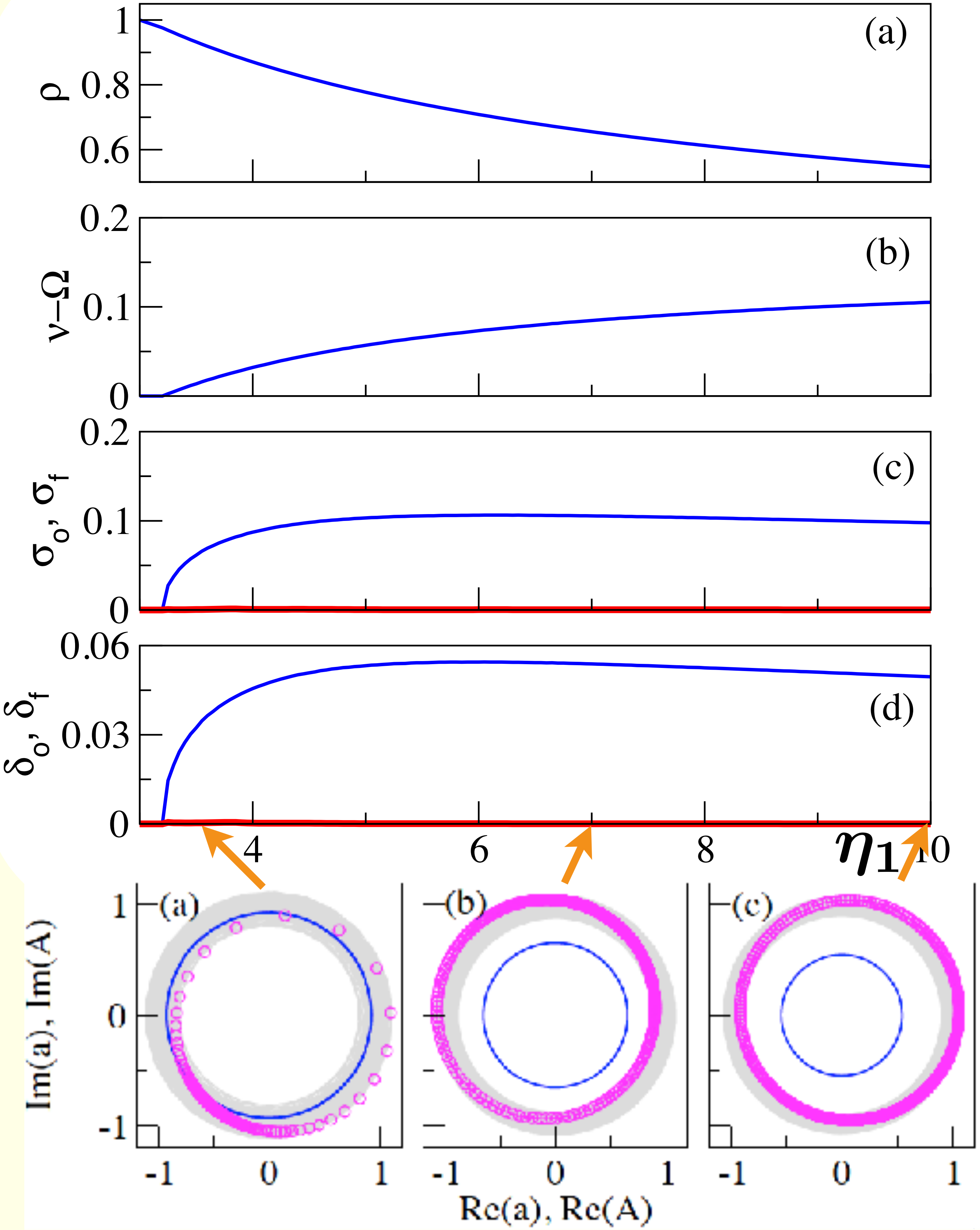
Transition for small $\varepsilon_2 = 0.3$

mean field amplitude

frequency difference

std of instantaneous frequency
(red: field, blue: oscillators)

std of instantaneous amplitude



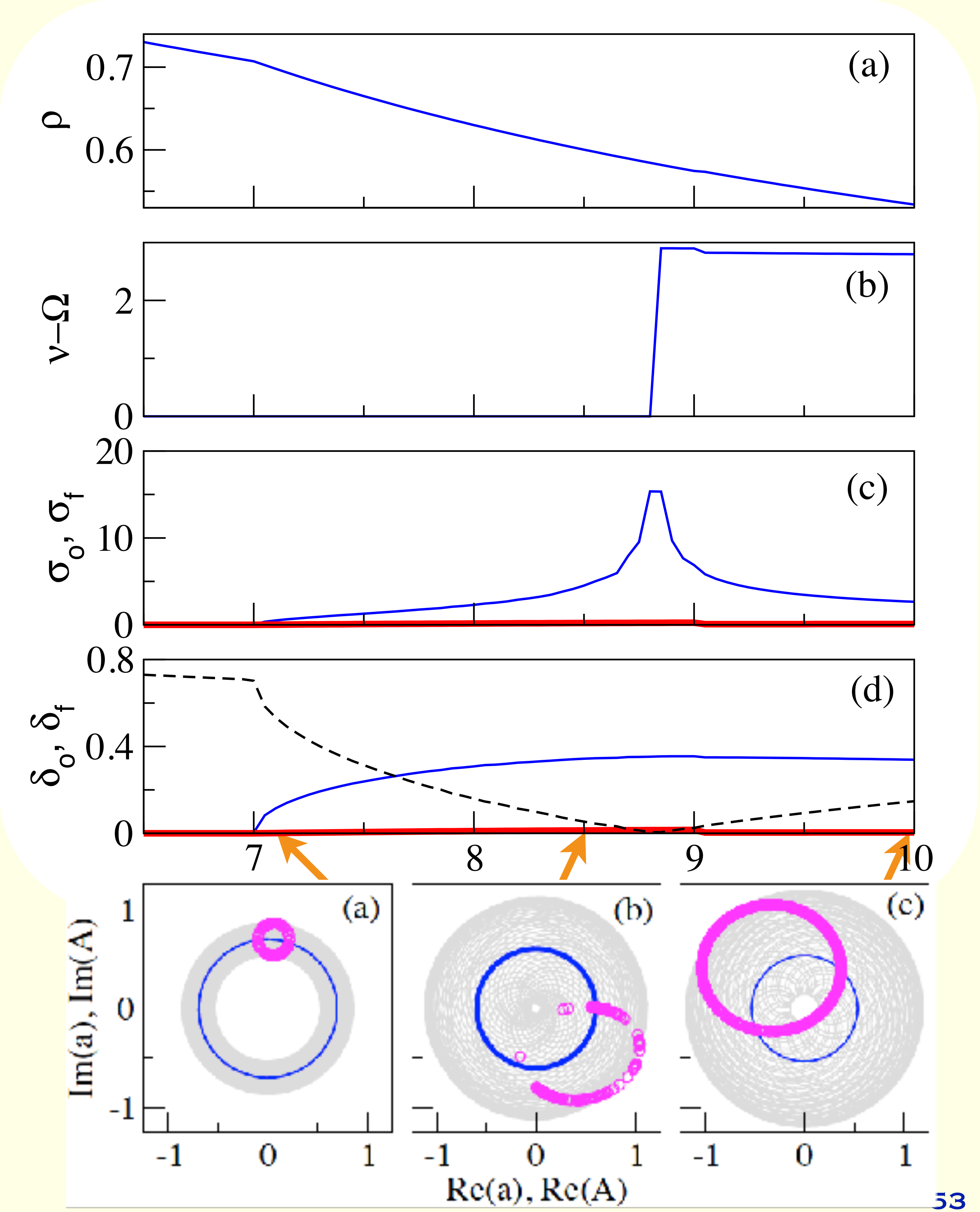
Transition for large $\varepsilon_2 = 3$

mean field amplitude

frequency difference

std of instantaneous frequency
(red: field, blue: oscillators)

std of instantaneous amplitude



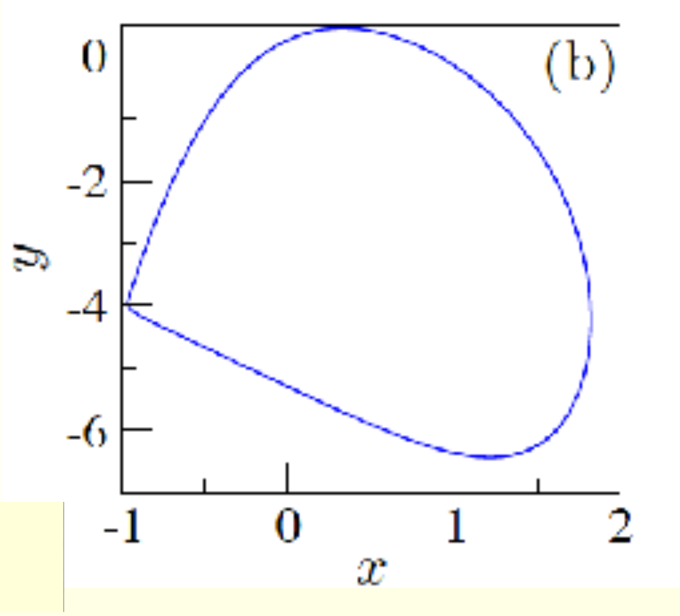
Globally coupled Hindmarsh-Rose neurons

$$\dot{x}_k = y_k - x_k^3 + 3x_k^2 - z_k + 5 + \varepsilon(X - x_k)$$

$$\dot{y}_k = 1 - 5x_k^2 - y_k$$

$$\dot{z}_k = 0.006\,[4(x_k + 1.56) - z_k]$$

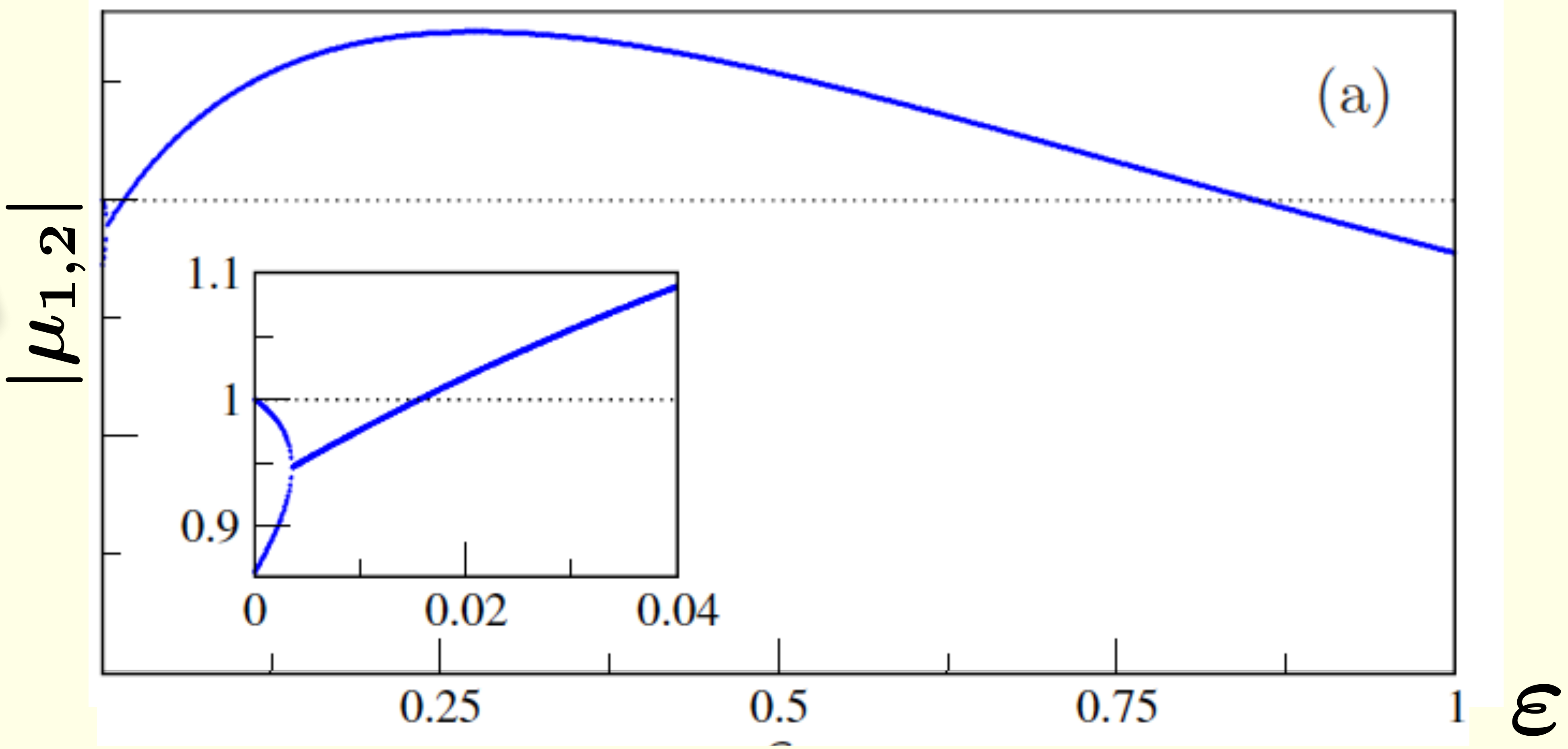
where mean field $X = N^{-1}\sum_j^N x_j$



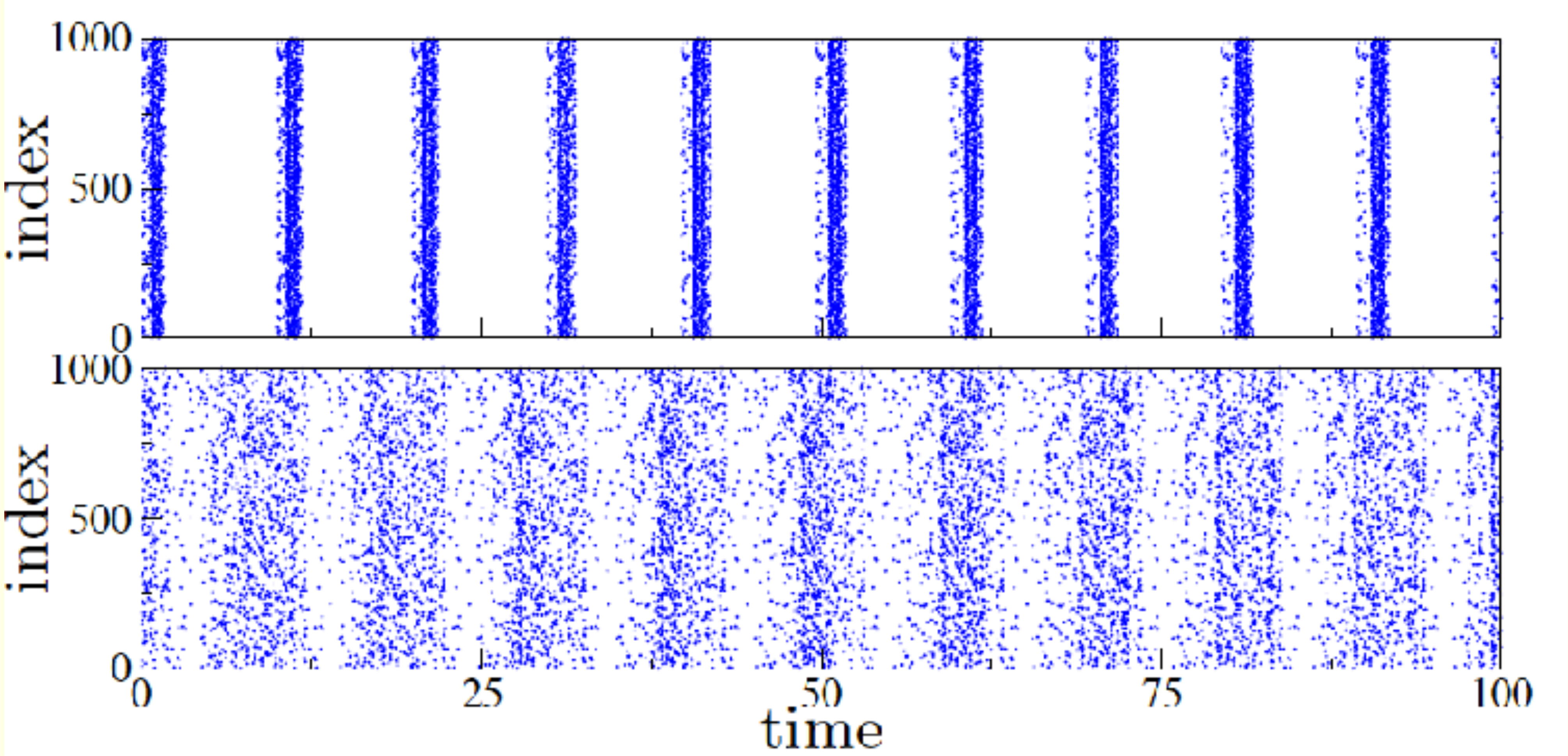
multipliers

$$|\mu| = e^{\lambda T}$$

transversal LE

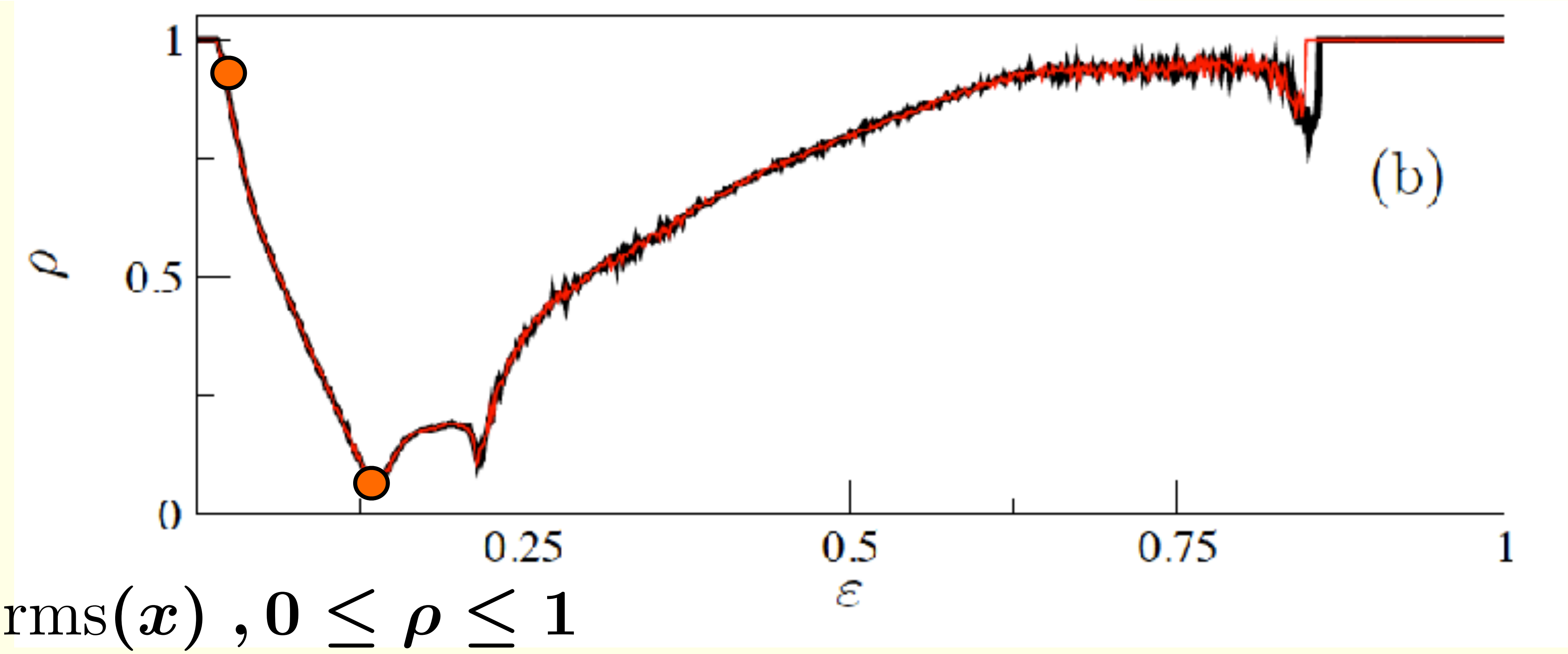


Globally coupled Hindmarsh-Rose neurons: results II

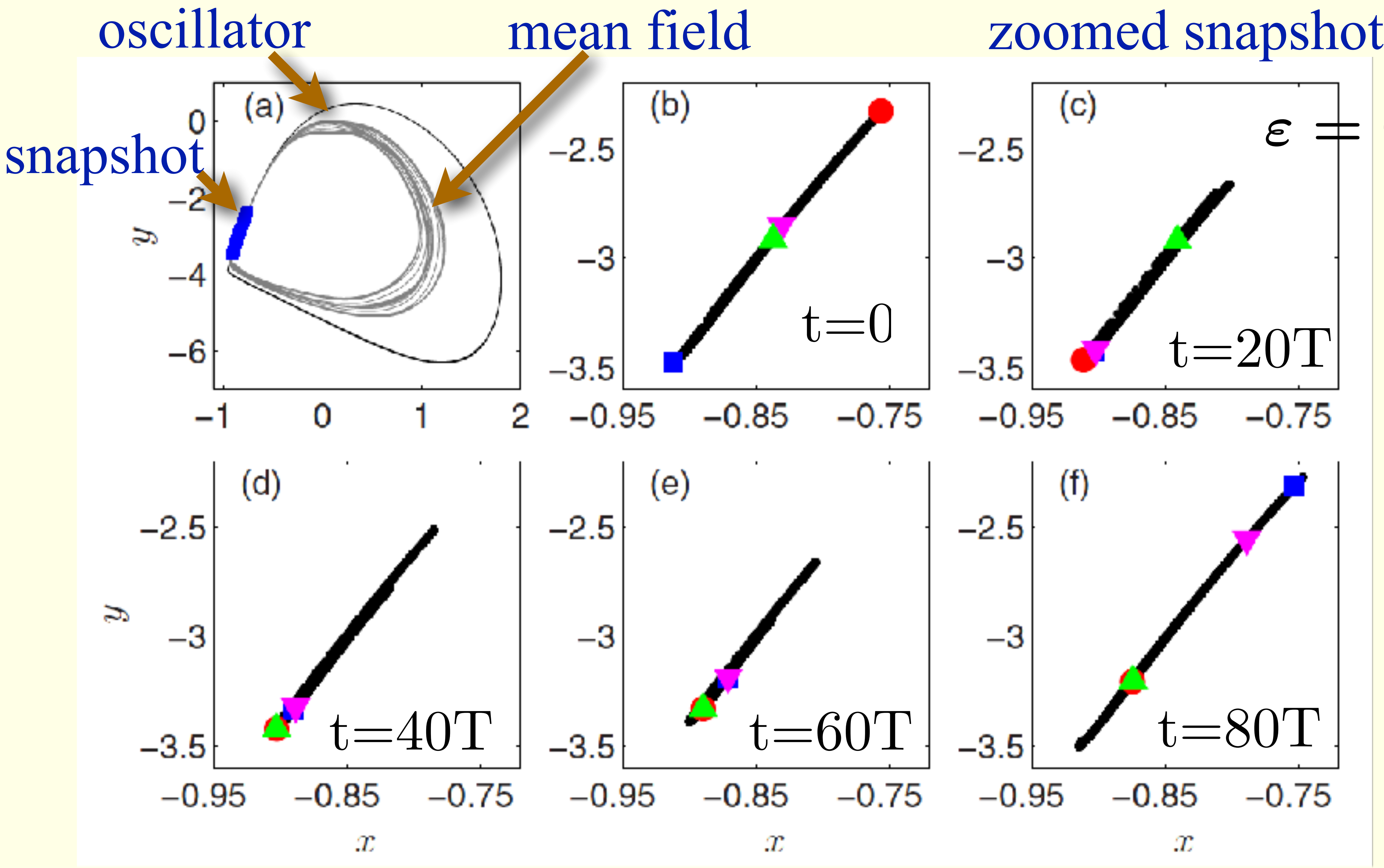


$\varepsilon = 0.04$

$\varepsilon = 0.13$



$\rho = \mathrm{rms}(\boldsymbol{X})/\mathrm{rms}(\boldsymbol{x})\ ,0 \leq \rho \leq 1$



oscillator

mean field

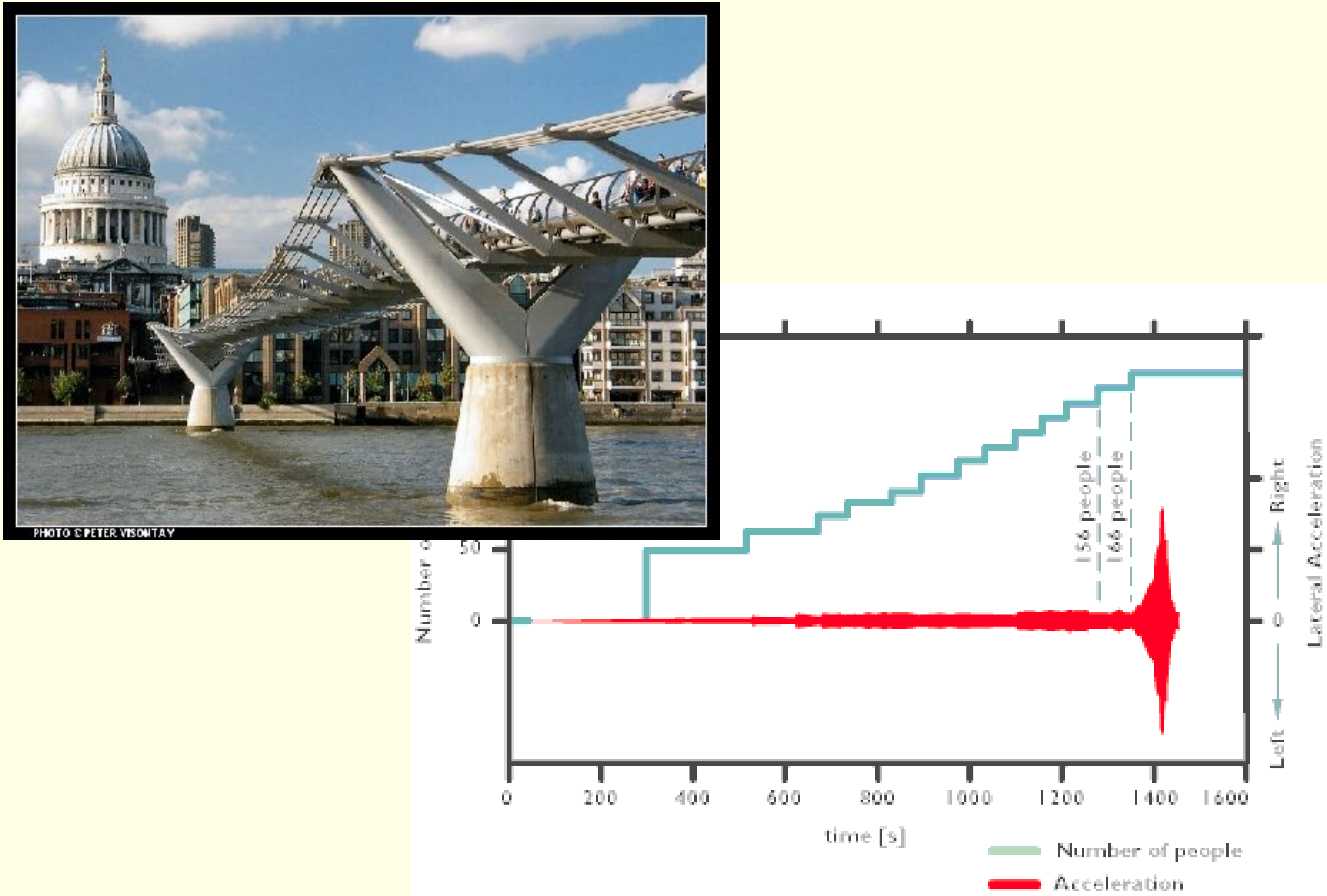
zoomed snapshot

snapshot

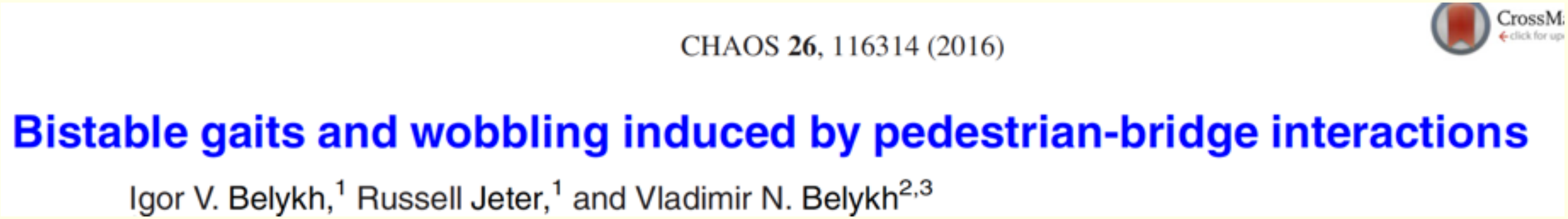
$\varepsilon = 0.04$

Different scenario of synchrony breaking (Hopf-like),
another type of quasiperiodic partial synchrony

Illustrative example of collective synchrony: The Millennium Bridge



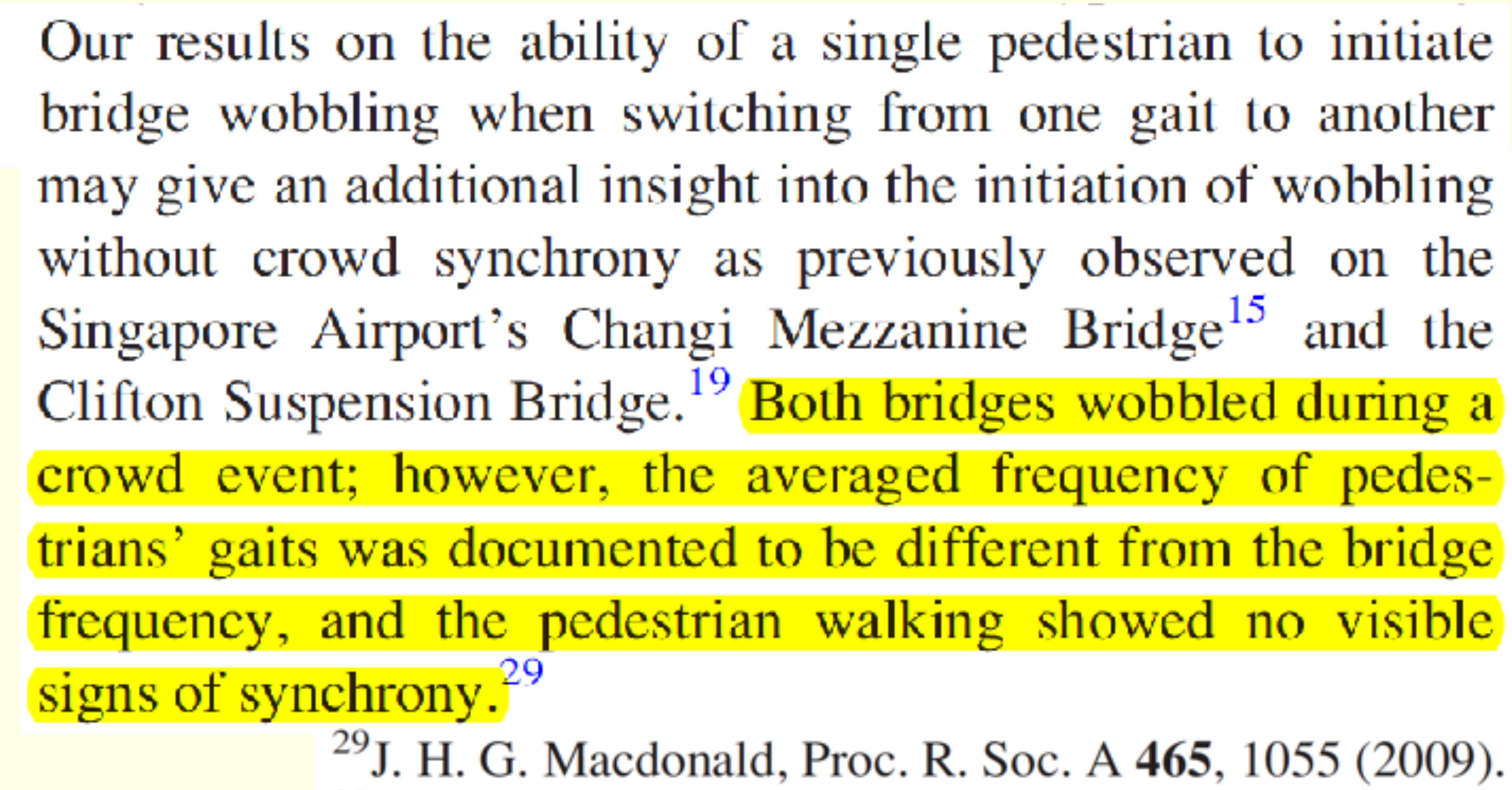
Bridge vibrations without synchrony



CHAOS **26**, 116314 (2016)

Bistable gaits and wobbling induced by pedestrian-bridge interactions

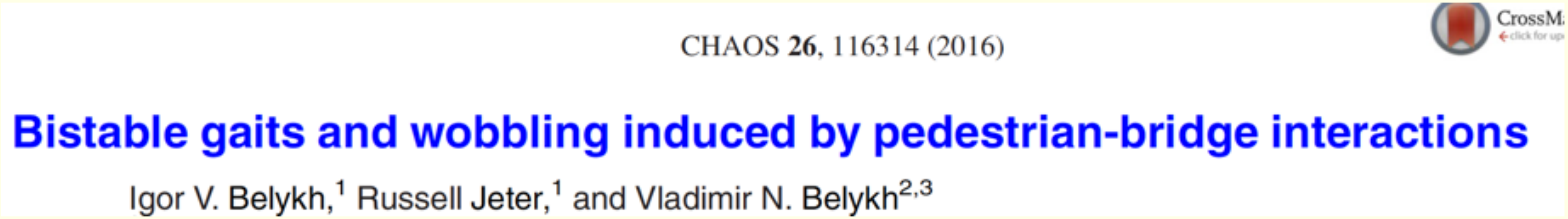
Igor V. Belykh,[1] Russell Jeter,[1] and Vladimir N. Belykh[2,3]



Our results on the ability of a single pedestrian to initiate bridge wobbling when switching from one gait to another may give an additional insight into the initiation of wobbling without crowd synchrony as previously observed on the Singapore Airport's Changi Mezzanine Bridge[15] and the Clifton Suspension Bridge.[19] Both bridges wobbled during a crowd event; however, the averaged frequency of pedestrians' gaits was documented to be different from the bridge frequency, and the pedestrian walking showed no visible signs of synchrony.[29]

[29] J. H. G. Macdonald, Proc. R. Soc. A **465**, 1055 (2009).

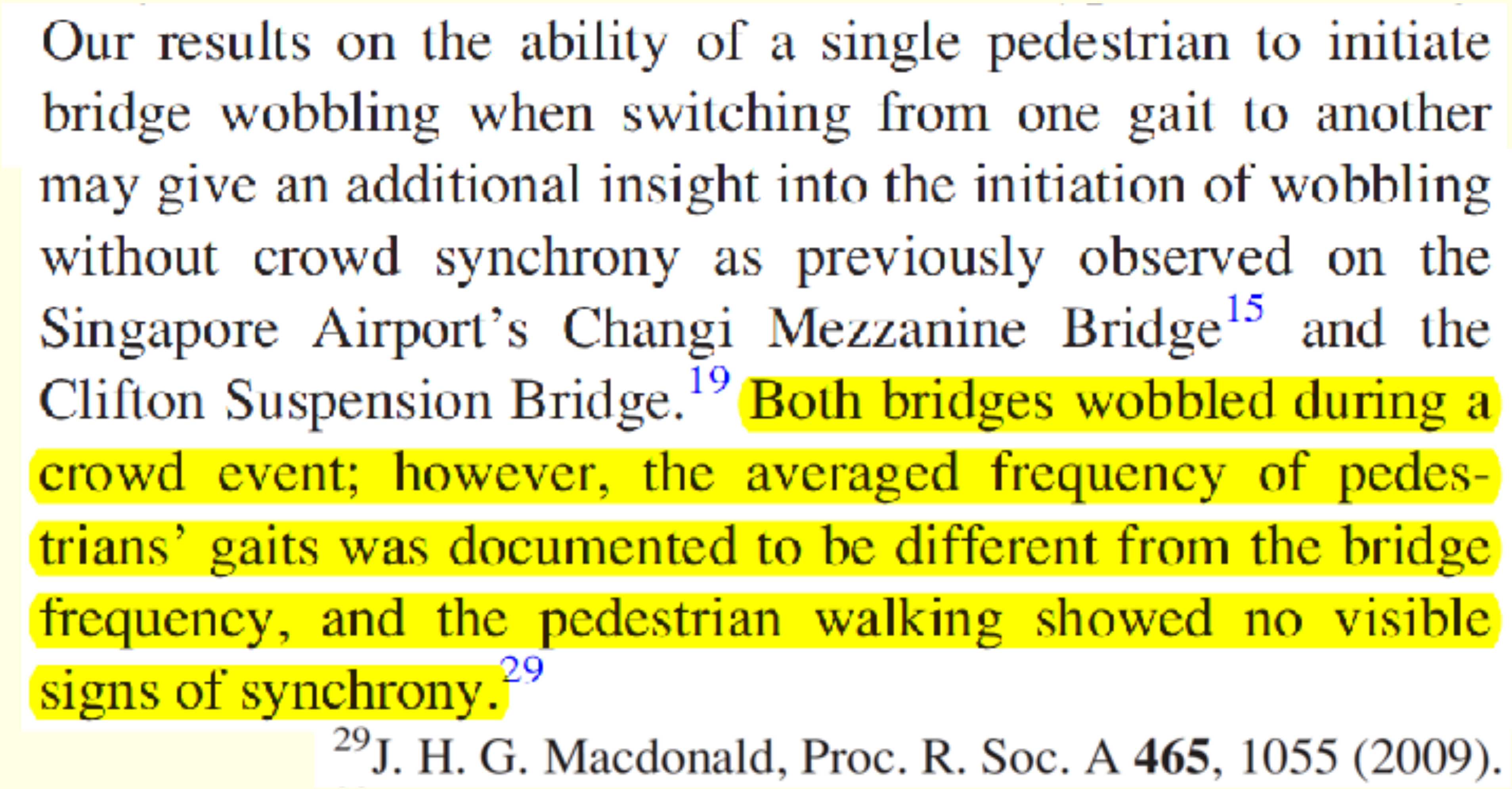
Bridge vibrations without synchrony



CHAOS **26**, 116314 (2016)

Bistable gaits and wobbling induced by pedestrian-bridge interactions

Igor V. Belykh,[1] Russell Jeter,[1] and Vladimir N. Belykh[2,3]



Our results on the ability of a single pedestrian to initiate bridge wobbling when switching from one gait to another may give an additional insight into the initiation of wobbling without crowd synchrony as previously observed on the Singapore Airport's Changi Mezzanine Bridge[15] and the Clifton Suspension Bridge.[19] Both bridges wobbled during a crowd event; however, the averaged frequency of pedestrians' gaits was documented to be different from the bridge frequency, and the pedestrian walking showed no visible signs of synchrony.[29]

[29] J. H. G. Macdonald, Proc. R. Soc. A **465**, 1055 (2009).

An example of quasiperiodic partial synchrony?

Conclusions

- Partially synchronous quasiperiodic dynamics appears at the border of stability of the synchronous state
- It appears in phase and full models, also for (weakly) inhomogeneous ensembles
- Further examples: van Vreeswijk model of coupled leaky integrate-and-fire neurons, …
- At least two non-trivial forms of quasiperiodicity
- Exact conditions for emergence of these states is not yet clear

References

1) P. Clusella, A. Politi, M. Rosenblum, *A minimal model of self-consistent partial synchrony*, New J. Physics 18 (2016) 093037

2) M. Rosenblum and A. Pikovsky, *Two types of quasiperiodic partial synchrony in oscillator ensembles*, PRE, **92,** 012919, 2015

3) A. Pikovsky and M. Rosenblum, *Dynamics of globally coupled oscillators: Progress and perspectives,* Chaos, **25,** 097616, 2015

COSMOS
Complex Oscillatory Systems:
Modeling and Analysis
Innovative Training Network
European Joint Doctorate

Thank you for your attention!