Inferring coupled oscillatory dynamics from observations

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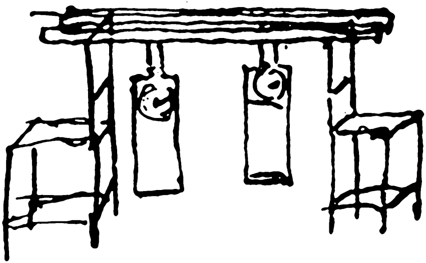
Outline

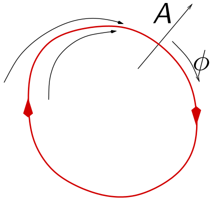
Coupled endogeneous (self-sustained) oscillators are described theoretically as a dynamical system for the oscillator's phases



We extract the dynamical equations from the non-invasively observed bi-variable data

Theoretical framework: Autonomous oscillator

- amplitude (form) of oscillations is fixed and st
- phase of oscillations is free



$\dot{\phi} = \omega_0$ (Lyapunov exp. 0)

$\dot{A} = -\gamma(A - A_0)$ (Lyapunov exp. $-\gamma$)

Theoretical framework: Autonomous oscillator

Phase is the variable proportional to the fraction of the period, it can be always chosen to rotate uniformly

Note: such a phase always exists and can be obtained from any cyclic variable θ by transformation

$$\phi = \omega_0 \int_0^{\theta} \left[\frac{d\theta}{dt} \right]^{-1} d\theta$$

Theoretical framework: Phase equations

One autonomous (self-sustained) oscillator

$$\dot{\varphi} = \omega$$

Coupled oscillators (pairwise coupling, first approximation)

$$\dot{\varphi}_k = \omega_k + \sum_j q_{jk}(\varphi_j, \varphi_k)$$

Term $q_{jk}(\varphi_j, \varphi_k)$ characterizes directional coupling $j \to k$.
If additionally frequency difference and coupling are small, one averages over the period

$$q_{jk}(\varphi_j, \varphi_k) \to Q_{jk}(\varphi_j - \varphi_k)$$

In many cases

$$q_{jk}(\varphi_j, \varphi_k) = \mathrm{PRC}(\varphi_k) \cdot \mathrm{Force}(\varphi_j)$$

Two coupled oscillators

Two uncoupled self-sustained oscillators:

$$\frac{d\varphi_1}{dt} = \omega_1 \qquad \frac{d\varphi_2}{dt} = \omega_2$$

Two weakly coupled oscillators:

$$\frac{d\varphi_1}{dt} = \omega_1 + q_{21}(\varphi_1, \varphi_2)$$
$$\frac{d\varphi_2}{dt} = \omega_2 + q_{12}(\varphi_1, \varphi_2)$$

The observed frequencies

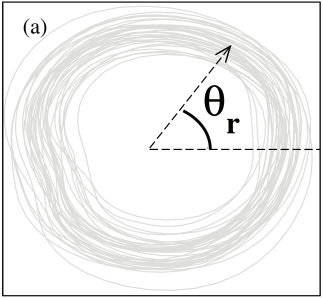
$$\Omega_1 = \left\langle \frac{d\varphi_1}{dt} \right\rangle \qquad \Omega_2 = \left\langle \frac{d\varphi_2}{dt} \right\rangle$$

deviate from the natural ones ω_1, ω_2

A scalar observable

Typically one observes a scalar quantity y that is a function of the system's state, $y = g(\mathbf{x})$, and records a scalar oscillatory time series
$Y = y(t_i)$
Using, e.g., the Hilbert transform $y \to \hat{y}$ one can obtain a two-dimensional embedding on the plane $(y,\ \hat{y})$
A **protophase** can be defined if the trajectory rotates around some point $(y_0,\ \hat{y}_0)$ in this (or other) embedding:

$$\theta = \text{arctan}\left(\frac{\hat{y} - \hat{y}_0}{y - y_0}\right)$$



Protophase vs the genuine phase

Because the protophase depends on the observable and the embedding, its dynamics generally differs from the dynamics of the genuine phase φ:

$$\dot{\varphi} = \omega \qquad\qquad \dot{\theta} = f(\theta)$$

$$\dot{\varphi}_1 = \omega_1 + q_{21}(\varphi_1, \varphi_2) \qquad\qquad \dot{\theta}_1 = f_{21}(\theta_1, \theta_2)$$

$$\dot{\varphi}_2 = \omega_2 + q_{12}(\varphi_1, \varphi_2) \qquad\qquad \dot{\theta}_2 = f_{12}(\theta_2, \theta_1)$$

Note: protophases provide same average frequencies, $\left\langle \dot{\theta} \right\rangle = \omega$
Hence, knowledge of θ suffices if we are only interested in detecting synchronization of two systems, but we want a more detailed description of interaction ...

Problems we address:

- Reconstruction of the genuine phase φ obeying $\dot{\varphi} = \omega$ from an observed protophase θ
- Reconstruction of the coupled equations for the genuine phases
  $\dot{\varphi}_1 = \omega_1 + q_{21}(\varphi_1, \varphi_2) \quad \dot{\varphi}_2 = \omega_2 + q_{12}(\varphi_1, \varphi_2)$
  from the observed bivariate data $\theta_{1,2}$
- Characterization of the coupling through properties of the coupling functions
- For reconstruction of the coupling network structure $\varphi_1 \leftrightarrow \varphi_2 \leftrightarrow \varphi_3 \ldots$ from the observed multivariate data θ_k see the talk by M. Rosenblum

From protophase toward the phase: one oscillator

Given: a time series $\Theta(t)$, $0 \leq t \leq T$
We look for a transformation $\theta \to \varphi$ satisfying

$$\frac{d\varphi}{d\theta} = \omega_0 \frac{dt}{d\theta}(\theta) = \sigma(\theta)$$

Averaging we obtain ($\sigma/2\pi$ is the probability density of θ)

$$\sigma(\theta) = 2\pi\langle\delta(\Theta(t) - \theta)\rangle = \frac{2\pi}{T}\int_0^T \delta(\Theta(t) - \theta)\, dt.$$

Using the Fourier transform of σ

$$\sigma(\theta) = \sum_n S_n e^{in\theta} \qquad S_n = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\theta) e^{-in\theta} d\theta$$

we get

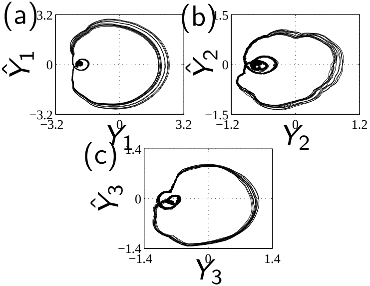
$$S_n = \frac{1}{T}\int_0^T e^{-in\Theta(t)} dt = \frac{1}{N}\sum_{k=1}^{N} e^{-in\Theta_k}$$

Final result: Transformation $\theta \to \varphi$ is

$$\varphi = \theta + \sum_{n\neq 0} \frac{S_n}{in}(e^{in\theta} - 1)$$

Example: Phase from the ECG I

We compute protophases Θ and phases Φ from three different channels (different leads) of the same ECG of a healthy male.
The Hilbert plane representations of these channels are:

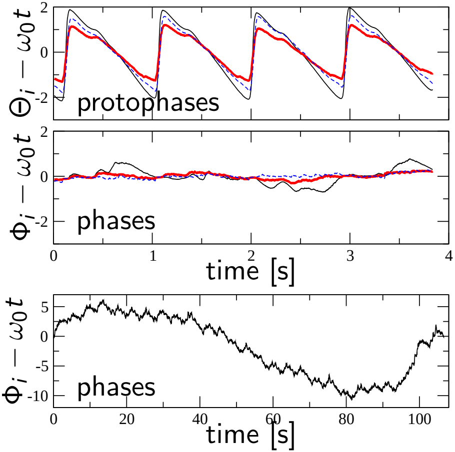


Because the usual angle variable is not monotonic, we estimate θ according to $\theta(t) = 2\pi \cdot l(t) \pmod{L}$, where $l(t)$ is the length along the trajectory in the Hilbert plane and L is the length of the loop

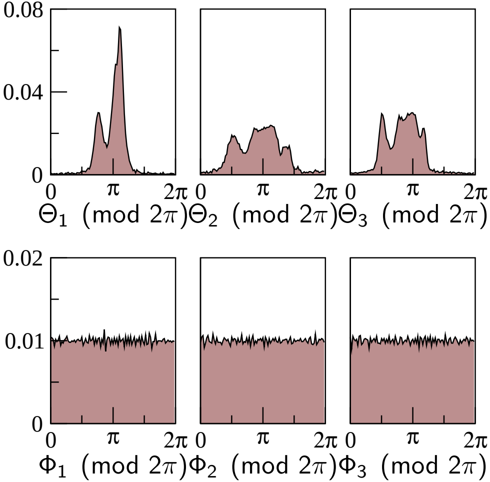
Example: Phase from the ECG II

The protophases and the phases obtained according to the procedure above for all channels.

The phases $\Phi_{1,2,3}(t)$ computed from three different observables nearly coincide and exhibit similar slow deviation from a linear growth, most likely due to the respiratory related rhythms.



Example: Phase from the ECG III



The effect of the transfromation $\Theta_i \rightarrow \Phi_i$ on the distributions of $\Theta_i \pmod{2\pi}$ and $\Phi_i \pmod{2\pi}$

Importance of protophase2phase transformation

Helpful for all statistical operations measuring phase interdependencies

- For synchronization index

$$\langle e^{i(n\varphi_1 - m\varphi_2)} \rangle$$

- For Kuramoto-Daido order parameters in ensemble

$$\langle e^{in\varphi} \rangle$$

For protophases these quantities do not vanish for independent phases
For the genuine phases these quantities do vanish for independent phases

Equations reconstruction

Given: bivariate protophase series $\Theta_1(t), \Theta_2(t)$, $0 \leq t \leq T$
First step: transform to the phases $\Phi_1(t), \Phi_2(t)$, $0 \leq t \leq T$
Second step: reconstruct equations

$$\dot{\varphi}_1 = f_{21}(\varphi_1, \varphi_2)$$
$$\dot{\varphi}_2 = f_{12}(\varphi_2, \varphi_1)$$

We represent the r.h.s. as Fourier series

$$\dot{\varphi}_1(\varphi_1, \varphi_2) = f_{21}(\varphi_1, \varphi_2) = \sum_{n,m} F_{nm} e^{in\varphi_1 + im\varphi_2}$$

and find F_{nm} from the minimum-of-the-error condition

$$\left\langle \left(\dot{\Phi}_1 - \sum_{n,m} F_{nm} e^{in\varphi_1 + im\varphi_2} \right)^2 \right\rangle \stackrel{!}{=} \min$$

Solution of the reconstruction problem

$$f_{21}(\varphi_1, \varphi_2) = \frac{b^{(1)}(\varphi_1, \varphi_2)}{c(\varphi_1, \varphi_2)} \qquad f_{12}(\varphi_1, \varphi_2) = \frac{b^{(2)}(\varphi_1, \varphi_2)}{c(\varphi_1, \varphi_2)}$$

where Fourier-coefficients $B_{n,m}^{(1,2)}$, $C_{n,m}$ are obtained by integrations of the time series

$$B_{n,m}^{(1)} = \frac{1}{T}\int_0^{\Phi_1(T)} d\Phi_1 \; e^{-in\Phi_1 - im\Phi_2} \qquad B_{n,m}^{(2)} = \frac{1}{T}\int_0^{\Phi_2(T)} d\Phi_2 \; e^{-in\Phi_1 - im\Phi_2}$$

$$C_{n,m} = \frac{1}{T}\int_0^{T} dt \; e^{-in\Phi_1(t) - im\Phi_2(t)}$$

Reconstructed phase equations

$$\frac{d\varphi_1}{dt} = \tilde{\omega}_1 + \tilde{q}_{21}(\varphi_1, \varphi_2)$$

$$\frac{d\varphi_2}{dt} = \tilde{\omega}_2 + \tilde{q}_{12}(\varphi_1, \varphi_2)$$

Functions $\tilde{q}^{(1,2)}$ are **observable-independent**, hence they are mostly suitable for characterizing strength and/or directionality of coupling

Reconstructed frequencies $\tilde{\omega}_{1,2}$ are not exactly natural ones, but can contain a non-oscillatory part of the coupling

Recovery of autonomous frequencies

Autonomous frequencies $\omega_{1,2}$ can be determined if we observe the systems for **at least two different values** of the coupling strength ε

Let us re-write the r.h.s of the equation for the 1st system as

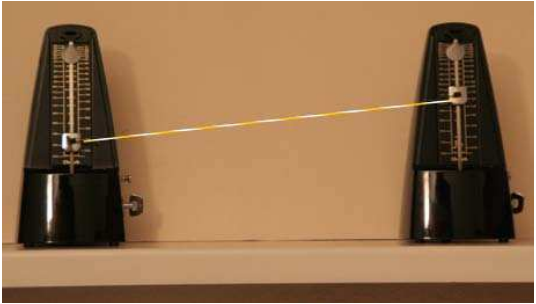
$$\dot{\varphi}_1 = \tilde{\omega}_1 + \tilde{q}^{(1)}(\varphi_1, \varphi_2) = \omega_1 + \varepsilon(q_0^{(1)} + Q^{(1)}(\varphi_1, \varphi_2))$$

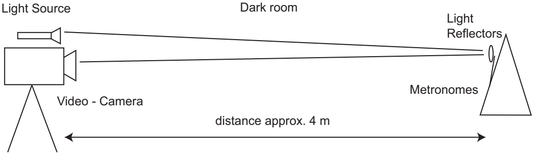
Our technique reconstructs the constant term $\tilde{\omega}_1 = \omega_1 + \varepsilon q_0^{(1)}$ and function $\tilde{q}_1 = \varepsilon Q^{(1)}(\varphi_1, \varphi_2)$

Suppose we have two measurements for $\varepsilon = \varepsilon'$ and $\varepsilon = \varepsilon''$ and recover $\tilde{\omega}_1' = \omega_1 + \varepsilon' q_0^{(1)}$, $\tilde{\omega}_1'' = \omega_1 + \varepsilon'' q_0^{(1)}$, $\tilde{q}_1' = \varepsilon' Q^{(1)}$, and $\tilde{q}_1'' = \varepsilon'' Q^{(1)}$. Then $\frac{\varepsilon'}{\varepsilon''} = \frac{||\tilde{q}_1'||}{||\tilde{q}_1''||}$ provides $\frac{\tilde{\omega}_1''-\omega_1}{\tilde{\omega}_1'-\omega_1} = \frac{||\tilde{q}_1'||}{||\tilde{q}_1''||}$

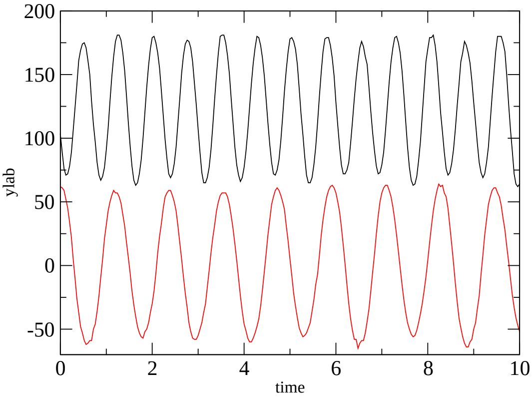
autonomous frequency ω_1

Experiment with coupled metronomes (Ralf Mrowka, Charité, Berlin)

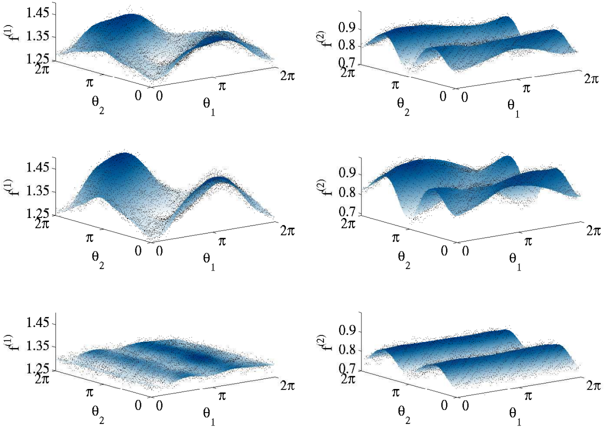




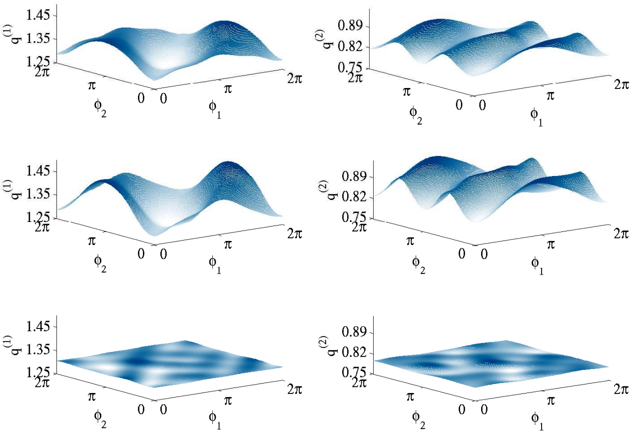
Experiment with coupled metronomes: raw data



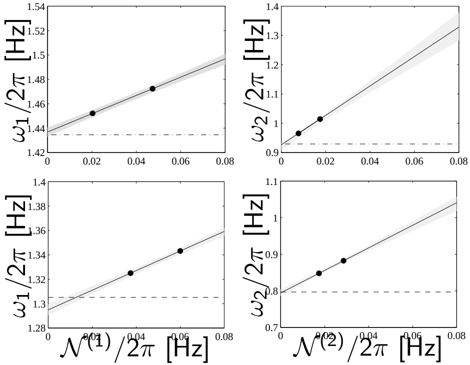
Experiment with coupled metronomes: protophases



Experiment with coupled metronomes: genuine phases



Experiment with coupled metronomes: recovering frequencies



Theoretical framework: Phase Response Curve

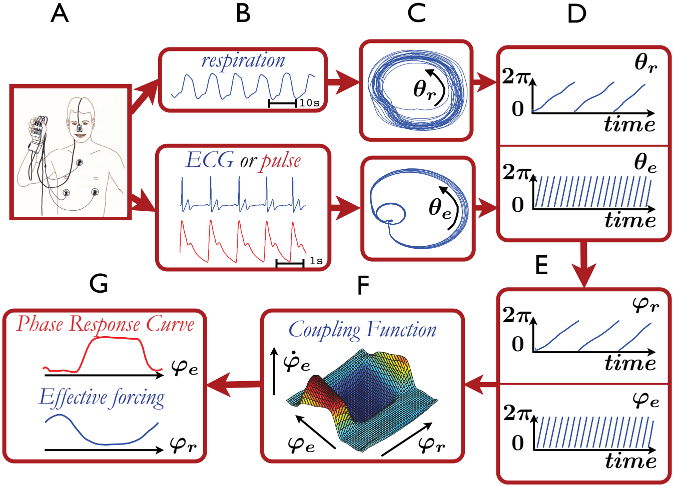
Theory suggests that for weak interaction

$$q_1(\varphi_1, \varphi_2) = Z_1(\varphi_1) I_2(\varphi_2) \,, \tag{1}$$

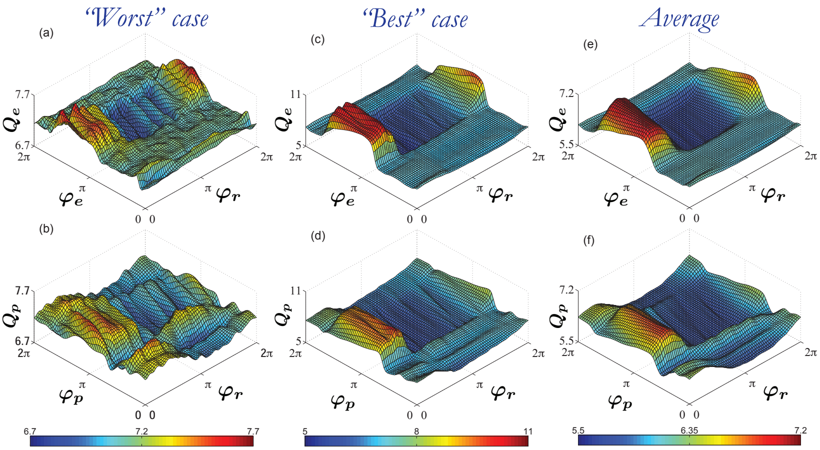
where $Z_1(\varphi_1)$ is the PRC of the first oscillator and $I_2(\varphi_2)$ is the forcing with which the oscillator 2 acts on 1, and similarly for $q_2 = Z_2(\varphi_2) I_1(\varphi_1)$.

Phase response curve from non-invasive observation of heart beats and respiration

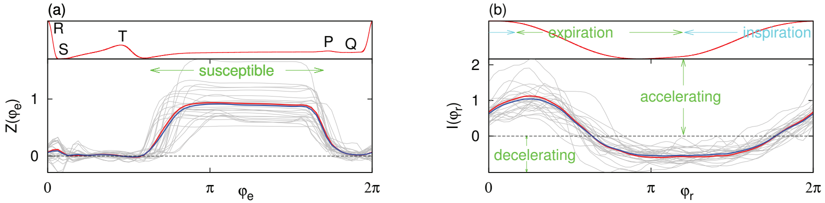
[Kralemann, Fühwirth, Pikovsky, Rosenblum, Kenner, Schaefer, Moser, Nature Communications, 4:2418 (2013)]



Result 1: Coupling functions for the human cardio-respiratory system: comparuing ECG and pulse



Result 2: Heart PRC for cardio-respiratory interaction.



Individual PRCs Z (a) and effective forcing I (b) for all ECG-based coupling functions (grey curves). Blue: the average over all individual (grey) curves. Red curves are obtained by decomposition of the averaged coupling function.

Result 3: Disentangling the heart rate into the respiratory-related component and the rest

Reconstructed ECG phase dynamics:

$$\dot{\varphi} = \omega + Q(\varphi, \varphi_r) + \xi(t)$$

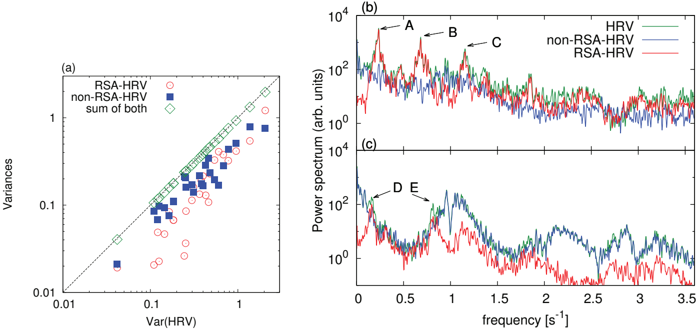
Respiratory-related heart phase dynamics:

$$\dot{\Phi} = \omega + Q(\Phi, \varphi_r)$$

Respiratory-free heart phase dynamics:

$$\dot{\Psi} = \omega + \xi(t)$$

Result of disentangling the heart rate into the respiratory-related component and the rest



original HRV $\dot{\varphi}_1(t)$ (green), the RSA-HRV component ($Q_1(t)$, red) and the non-RSA-HRV component ($\dot{\varphi}_1 - Q_1$, blue)

Conclusion

- A technique to recover **genuine phases** from noisy oscillatory observables
- This preprocessing is a necessary step prior to statistical analysis of phases (e.g. prior to synchronization index calculations)
- Reconstruction of **observable-independent equations** of phase dynamics provides **complete description of interacting systems** within the phase approximation
- Autonomous frequencies from observations of coupled systems
- **Disentangling different components** in the phase dynamics
- MATLAB package available: http://www.agnld.uni-potsdam.de/∼mros/damoco2.html