Chimera and other complex states in networks of coupled oscillators

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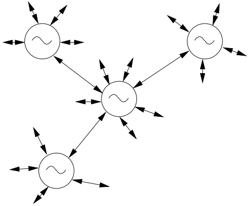
Contents

Ensembles of globally (all-to-all) couples oscillators

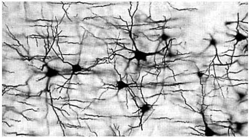
- Physics: arrays of Josephson junctions, multimode lasers,spin-torque oscillators,...
- Biology and neuroscience: cardiac pacemaker cells, population of fireflies, neuronal ensembles,...
- Social behavior: applause in a large audience, pedestrians on a bridge,...



Main effect: Synchronization

Mutual coupling adjusts phases of individual systems, which start to keep pace with each other

Synchronization can be treated as a nonequilibrium phase transition!



Kuramoto-Sakaguchi model: coupled phase oscillators

Phase oscillators with all-to-all pair-wise coupling

$$\dot{\varphi}_k = \omega + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k + \beta)$$
$$= \omega_k + \varepsilon R(t) \sin(\Theta(t) - \varphi_k + \beta) = \omega_k + \varepsilon \mathrm{Im}(Z e^{i\beta} e^{-i\varphi_k})$$

System can be written as a mean-field coupling with the mean field (complex order parameter Z)

$$Z = Re^{i\Theta} = \frac{1}{N} \sum_k e^{i\varphi_k}$$

Identical oscillators:

- synchronization for attractive coupling $\varepsilon \cos \beta > 0$: $|Z| = 1$
- desynchronization for repulsive coupling $\varepsilon \cos \beta < 0$: $|Z| = 0$

Closed description of the collective mode

Watanabe and Strogatz (1994), Ott and Antonsen (2008)
For simplicity, we consider the thermodynamic limit $N \to \infty$ only
Identical oscillators driven by the common complex field H

$$\dot{\varphi}_k = \omega + \mathrm{Im}(He^{-i\varphi_k})$$

Order parameter $Z = \langle e^{i\varphi} \rangle$ obeys a dynamical equation

$$\frac{dZ}{dt} = i\omega Z + \frac{1}{2}(H - H^* Z^2)$$

In the Kuramoto-Sakaguchi case the driving field is $H = \varepsilon Z e^{i\beta}$, thus

$$\frac{dZ}{dt} = i\omega Z + \frac{\varepsilon}{2}(Ze^{i\beta} - e^{-i\beta}|Z|^2 Z)$$

Synchronized ensemble as an effective collective oscillator

$$\frac{dZ}{dt} = i\omega Z + \frac{\varepsilon}{2}(e^{i\beta} - e^{-i\beta}|Z|^2)Z$$

Equation for the "amplitude" $R = |Z|$:

$$\frac{dR}{dt} = \frac{\varepsilon}{2}\cos\beta(R - R^3)$$

$\cos\beta > 0$: Stable synchrony $R \to 1$
$\cos\beta = 0$: Neutral (conservative) case
$\cos\beta < 0$: Stable full asynchrony $R \to 0$

Ensemble with distribution of frequencies as an effective collective oscillator with damping

Lorentzian distribution of frequencies with width Δ and mean frequency ω:

$$\frac{dZ}{dt} = i\omega Z - \Delta Z + \frac{\varepsilon}{2}(e^{i\beta} - e^{-i\beta}|Z|^2)Z$$

Classical Kuramoto case: $\beta = 0$:

$$\frac{dZ}{dt} = i\omega Z + (\frac{\varepsilon}{2} - \Delta)Z - \frac{\varepsilon}{2}|Z|^2)Z$$

Critical coupling $\varepsilon_c = 2\Delta$ separates disordered ($|Z| = 0$) and partially synchronized ($0 < |Z| < 1$) regimes

Conclusion to Ott-Antonsen theory

- Population of coupled oscillators can be fully described by a complex order parameter
- Dynamics of this order parameter is like that of a complex amplitude at a Hopf bifurcation
- From the viewpoint of collective dynamics, the population is like one effective oscillator

Chimera states in setups with symmetric coupling

- Nonlocal coupling in a spatially extended situation

$$\dot{\varphi}(x) = \omega + \varepsilon \mathrm{Im}(H(x) e^{-i\varphi(x)}) \qquad H(x) = e^{i\beta} \int dy\, K(x-y) e^{i\varphi(y)}$$

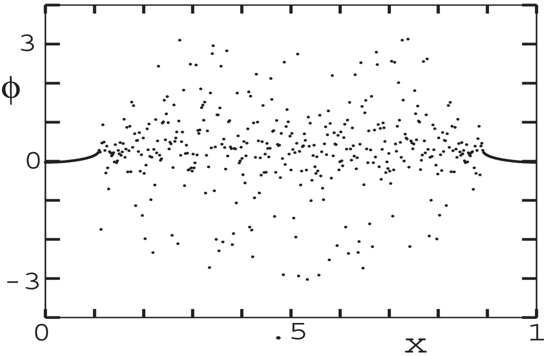
- Two coupled populations

$$\dot{\varphi}_k^a = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^a + \beta) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^a + \beta)$$

$$\dot{\varphi}_k^b = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^b + \beta) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^b + \beta)$$

Chimera states

Y. Kuramoto and D. Battogtokh observed in 2002 a symmetry breaking in non-locally coupled oscillators
$H(x) = e^{i\beta} \int dx' \exp[-|x' - x|] \exp[i\varphi(x')]$



This regime was called "chimera" by Abrams and Strogatz

Chimera in two subpopulations

Model by Abrams et al:

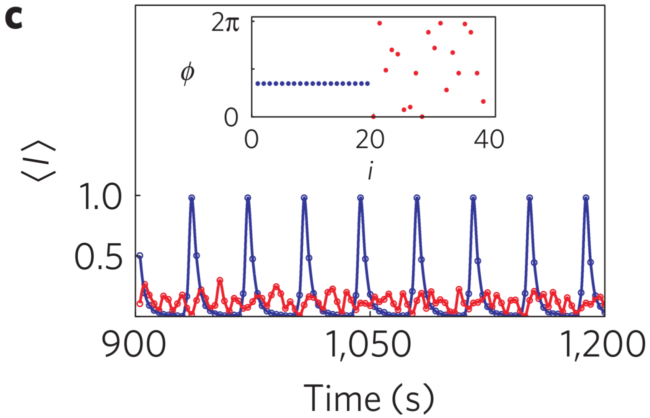
$$\dot{\varphi}_k^a = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^a + \beta) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^a + \beta)$$

$$\dot{\varphi}_k^b = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^b + \beta) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^b + \beta)$$

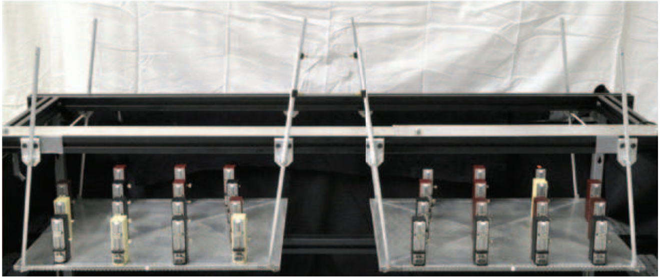
can be reduced to two coupled collective modes!
$|Z^a| = 1$ and $|Z^b|(t) < 1$ quasiperiodic are observed

Chimera in experiments I

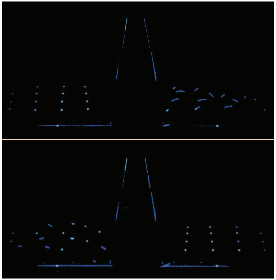
Tinsley et al: two populations of chemical oscillators



Chimera in experiments II



Erik A. Martens,
MPI für Dynamik und Selbstorganisation



Using Ott-Antonsen ansatz to describe chimera

Thermodynamic limit (oscillatory medium)
In terms of phases:

$$\partial_t \phi = \omega + \mathrm{Im}\left(e^{-i(\phi+\alpha)} \int \kappa e^{-|x-\tilde{x}|} e^{i\phi(\tilde{x},t)} d\tilde{x} \right),$$

In terms of local coarse-grained order parameter field

$$Z(x,t) = \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} e^{i\phi(\tilde{x},t)} d\tilde{x}$$

we have

$$\partial_t Z = i\omega Z + \left(e^{-i\alpha} H - e^{i\alpha} H^* Z^2 \right)/2$$

$$H(x,t) = \int \kappa e^{-|x-\tilde{x}|} Z(\tilde{x},t)\, d\tilde{x}$$

Physical interpretation of exponential coupling kernel

Field H can be treated as an extra diffusive mediator field which transfers coupling:

$$\tau \partial_t H = \kappa^{-2} \partial^2_{xx} H - H + Z \ .$$

In the limit $\tau \to 0$ this field obeys

$$\partial^2_{xx} H - \kappa^2 H = -\kappa^2 Z$$

and is described by the exponential kernel

$$H(x,t) = \int \kappa e^{-|x-\tilde{x}|} Z(\tilde{x},t) \, d\tilde{x}$$

Chimera state as a periodic pattern

$$\partial_t Z = i\omega Z + \frac{e^{-i\alpha} H - e^{i\alpha} H^* Z^2}{2} \qquad \partial^2_{xx} H - H = -Z$$

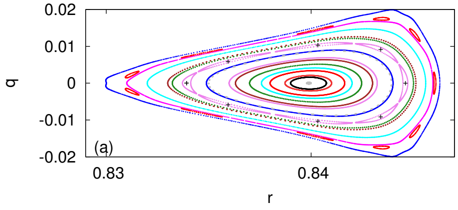
Uniformly rotating chimera state

$$Z(x,t) = z(x) e^{i(\omega+\Omega)t}, \qquad H(x,t) = h(x) e^{i(\omega+\Omega)t}$$

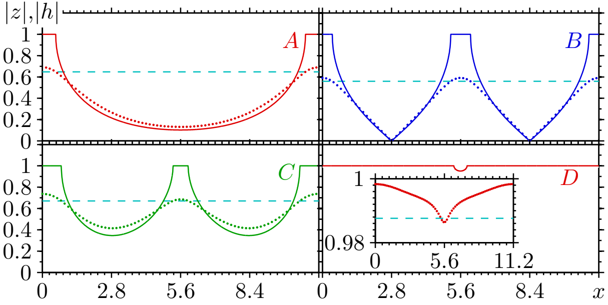
can be described as a stationary pattern in the system "complex ODE+algebraic equation"

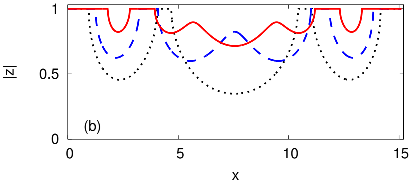
$$e^{i\alpha} h^* z^2 + 2i\Omega z - e^{-i\alpha} h = 0 \;, \qquad \frac{d^2}{dx^2} h - h = -z$$

Nonlinear reversible system of 3 equations - 2-dimensional quasi-Hamiltonian map



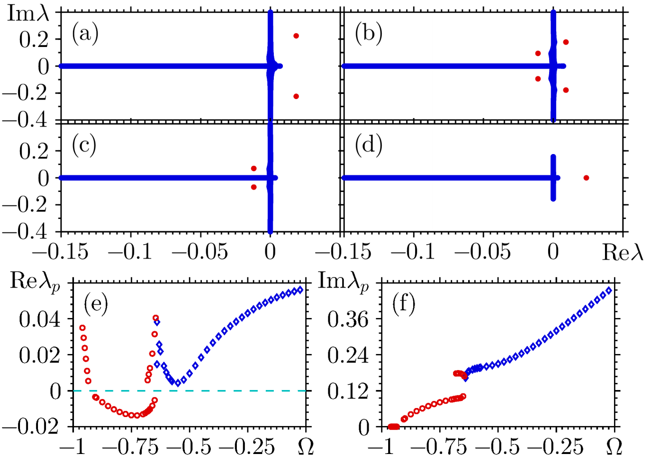
Simple and complex chimera patterns





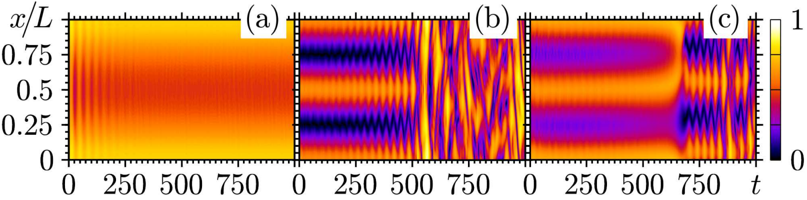
Mostly difficult issue: stability

Essential spectrum: purely real and purely imaginary eigenvalues
Discrete spectrum: responsible for instability



only simplest chimera patterns appear to be stable

Unstable patterns result in turbulence

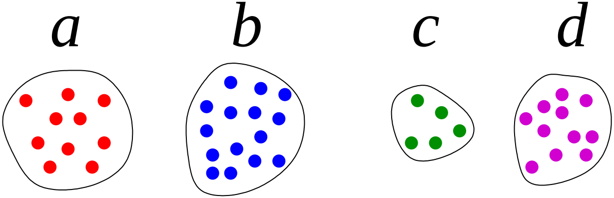


Conclusions to chimera states

- Population of coupled oscillators can be treated in terms of collective fields as a nonlinear “macro-oscillator”
- Spatial coupling reduces to a pattern-forming system, chimera states as non-trivial spatial patterns
- Future work: synchronization waves?

Hierarchically organized populations of oscillators

We consider populations consisting of M subgroups (of different sizes)



Each subgroup is described by WS-OA equations

$\Rightarrow$ system of coupled equations completely describes the ensemble

Many populations = many collective modes

$$\frac{dZ_a}{dt} = (i\omega_a - \Delta_a)Z_a + \frac{1}{2}(H_a - H_a^* Z_a^2)$$

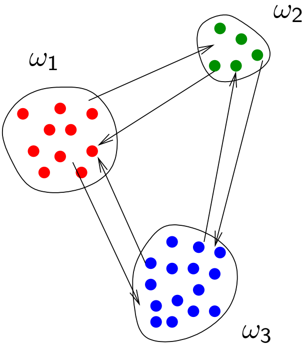
General force acting on subgroup a:

$$H_a = \sum_{b=1}^{M} n_b E_{a,b} Z_b + F_{ext,a}(t)$$

n_b: relative subgroup size
$E_{a,b}$: coupling between subgroups a and b

Multifrequency I: Resonantly interacting ensembles

[M. Komarov and A. P., Phys. Rev. Lett. 110, 134101 (2013)]



Most elementary nontrivial resonance $\omega_1 + \omega_2 = \omega_3$

Hypernetwork of coupled oscillators

On the level of individual oscillators (phase ϕ from ω_1, phase ψ from ω_2, phase θ from $\omega_3 = \omega_2 + \omega_1$) one has to take into account **triple interactions**:

$$\dot{\phi}_k = \ldots + \Gamma_1 \sum\nolimits_{m,l} \sin(\theta_m - \psi_l - \phi_k + \beta_1)$$
$$\dot{\psi}_k = \ldots + \Gamma_2 \sum\nolimits_{m,l} \sin(\theta_m - \phi_l - \psi_k + \beta_2)$$
$$\dot{\theta}_k = \ldots + \Gamma_3 \sum\nolimits_{m,l} \sin(\phi_m + \psi_l - \theta_k + \beta_3)$$

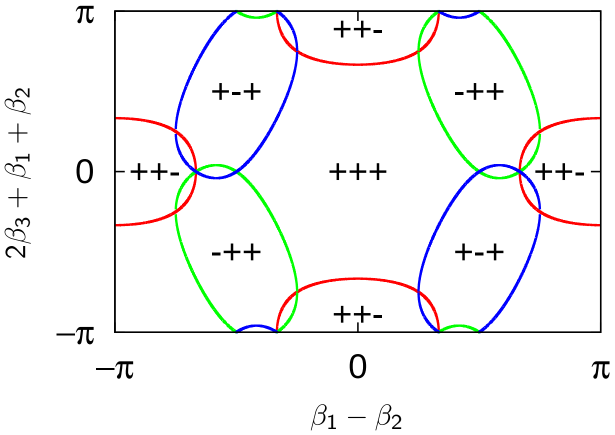
Hypernetwork: triple or multiple interactions

Set of three OA equations

On the level of effective oscillators describing order parameters, one has a triplet of Stuart-Landau equations with resonant coupling terms

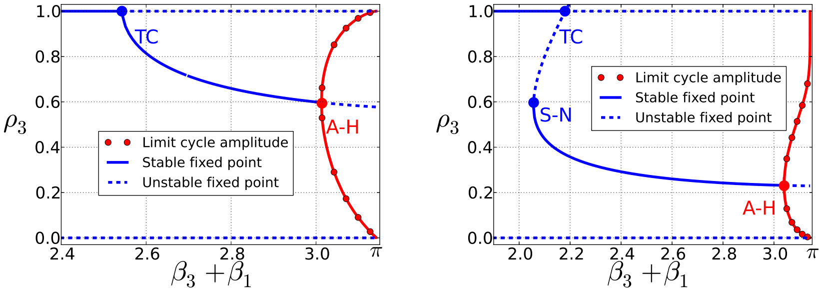
$$\dot{Z}_1 = Z_1(i\omega_1 - \delta_1) + (\epsilon_1 Z_1 + \gamma_1 Z_2^* Z_3 - Z_1^2(\epsilon_1^* Z_1^* + \gamma_1^* Z_2 Z_3^*))$$
$$\dot{Z}_2 = Z_2(i\omega_2 - \delta_2) + (\epsilon_2 Z_2 + \gamma_2 Z_1^* Z_3 - Z_2^2(\epsilon_2^* Z_2^* + \gamma_2^* Z_1 Z_3^*))$$
$$\dot{Z}_3 = Z_3(i\omega_3 - \delta_3) + (\epsilon_3 Z_3 + \gamma_3 Z_1 Z_2 - Z_3^2(\epsilon_3^* Z_3^* + \gamma_3^* Z_1^* Z_2^*))$$

Regions of synchronizing and desynchronizing effect from triple coupling

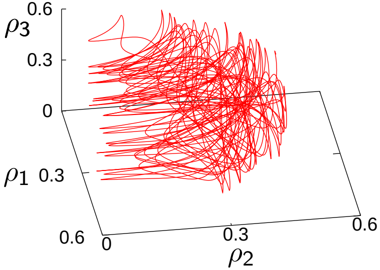


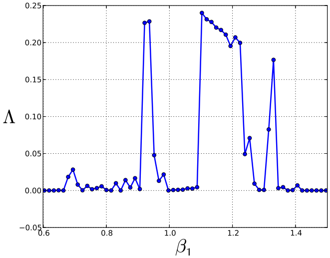
Bifurcations in dependence on phase constants

Different transitions from full to partial to oscillating synchrony



Chaos of order parameters





Multifrequency II: Non-resonantly interacting ensembles

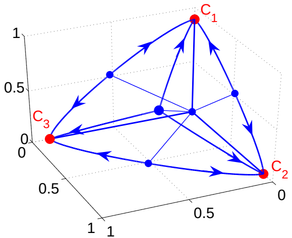
[M. Komarov, A. P., Phys. Rev. E, v. 84, 016210 (2011)]

Frequencies are different – all interactions are non-resonant
Only amplitudes of the order parameters can be involved in the coupling between subpopulations
General equations are of type

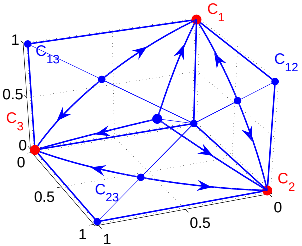
$$\dot{R}_l = (-\Delta_l - \Gamma_{lm} R_m^2) R_l + (a_l + A_{lm} R_m^2)(1 - R_l^2) R_l, \qquad l = 1, \ldots, L$$

where Γ_{lm} and A_{lm} decsribe the coupling

Competition for synchrony



(a)

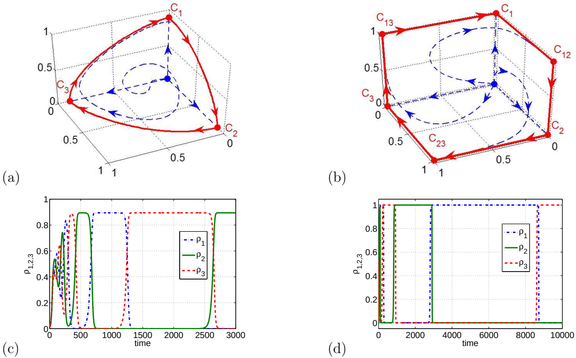


(b)

Only one ensemble is synchronous – depending on initial conditions

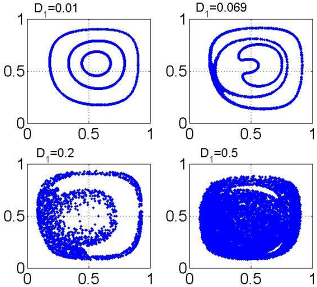
Heteroclinic synchrony cycles

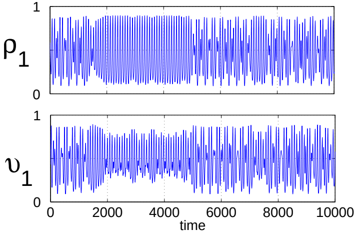
Sequential synchrony (partial or full) in populations



Chaotic synchrony cycles

Order parameters demonstrate chaotic oscillations





Conclusions to multifrequency populations

- Closed description with macroscopic equations for global modes
- Triple interaction – hypernetwork organization
- Competition for synchrony
- Heteroclinic cycles and chaotic states